









Modelling Financial Markets

Chapter 1: Stylised Facts and Returns



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Which Regularities Do Returns Share Across Markets?



Benoît Mandelbrot. **Photo: Rama (2007); CC BY-SA 2.0 FR; Wikimedia Commons**

"... large changes tend to be followed by large changes—of either sign—and small changes tend to be followed by small changes ..."

— *Mandelbrot (1963)*

- S&P 500 (Standard & Poor's) daily log returns, 1990–2026: lag-1 autocorrelation -0.08
  - only 7 of 100 lags leave the robust band
- Autocorrelation of absolute returns $|r_t|$: 0.27 at lag 1 and still 0.09 at lag 100
- Excess kurtosis of the S&P 500: 10.9 for daily returns, 1.6 for 60-day returns
  - the Normal distribution is still rejected at 60 days ($z = 4.1$)
- Long-memory parameter of $|r_t|$, $\hat{d}$: between 0.35 and 0.50 in all five markets

▸ Central chart: Uncorrelated but Not Independent

Unpredictable Returns, Predictable Volatility

What you should be able to do

- compute and aggregate simple and log returns
- compute moments, quantiles and the Hill estimator
- interpret zero correlation: not independence
- interpret tests with valid reference distributions
- compute volatility from the open, high, low and close
- interpret a hypothesis of an artificial intelligence (AI) assistant
- Further Topics and Appendix: reading for familiarity

Definition

- A **stylised fact** is a qualitative statistical regularity of **returns**
  - seen across assets, markets and periods Cont (2001)
- Each stylised fact remains a hypothesis, re-tested on each market and frequency Ratliff-Crain, Van Oort, Koehler & Tivnan (2025)
- Key message: returns show little linear autocorrelation
  - but their **size** is highly predictable

- Stylised facts are the benchmark every model of returns must reproduce
  - proofs: Appendix
  - the i.i.d. (independent and identically distributed) Normal-distribution model fails most of them

The Eleven Stylised Facts

Group	Facts
Distribution	2. heavy tails; 3. gain/loss asymmetry; 4. aggregational Gaussianity; 7. conditional heavy tails
Dependence of returns	1. absence of linear autocorrelation
Dependence of volatility	5. intermittency; 6. volatility clustering; 8. slow decay of the autocorrelation of $\lvert r_t \rvert$; 11. asymmetry in time scales
Links to other variables	9. leverage effect; 10. volume/volatility correlation

- The list of eleven facts is that of Cont (2001)
- On five markets, with data up to September 2026, we re-test facts 1–6 and 8–10
  - we add the inference each test needs; facts 7 and 11 need a volatility model (Chapters 5 and 9)
- Early evidence: Mandelbrot (1963), Fama (1965)
- Textbook treatment: Taylor (2008), Franke, Härdle & Hafner (2019)
- Detailed examples and advanced topics are in Further Topics, after the Conclusions
  - the footer button leads there; the Back button returns to the starting slide

Five Markets with Different Quoting Rules and Calendars

Series	Price used	Type, currency	Calendar	Sample	T
S&P 500	daily close (plus open, high, low)	price index, USD	US trading days	1990–2026	9 245
BET-TR	daily close	total return, RON	trading days of the Bucharest Stock Exchange (BVB)	Sep 2014 – 2026	2 992
Bitcoin	daily close	price, USD	every day, 24/7	Sep 2014 – 2026	4 384
EUR/RON	reference rate of the National Bank of Romania (BNR)	RON per EUR, managed float	Romanian working days	Jul 2005 – 2026	5 352
Gold	daily close	spot, USD per ounce	weekdays only	1990–2026	9 559

- Daily closes up to 18 September 2026
  - log returns on each series' own calendar
  - T: the number of returns
- BET-TR: the BET index (*Bucharest Exchange Trading*) of the BVB (Bursa de Valori București, the Bucharest Stock Exchange), with dividends reinvested (total return)
- **Managed float**: a market-determined exchange rate with central-bank interventions
  - we use the BNR (Banca Națională a României, the National Bank of Romania) reference rate: in several years the EUR (euro)/RON (euro/leu) interbank quotes had a lag-1 autocorrelation of -0.4 to -0.7, which came from quote errors

Simple or Log Return: What Adds Up over Time and What Adds Up across Assets

	Simple return R_t	Log return r_t
Definition	$(P_t + D_t)/P_{t-\mathbf{1}} - 1$	$\ln(1+R_t) = \ln P_t - \ln P_{t-\mathbf{1}}$ (no dividend)
Over k periods	$\prod_{j=\mathbf{0}}^{k-\mathbf{1}}(1+R_{t-j}) - 1$	$\sum_{j=\mathbf{0}}^{k-\mathbf{1}} r_{t-j}$: additive over time
Across assets	$R_{p,t} = \sum_i w_i R_{i,t}$: exact	only approximately
Range	$R_t \geq -100\%$	$(-\infty, +\infty)$
Use	portfolios, performance reporting	single-asset dynamics, central limit theorem (CLT)

- Notation
  - P_t: the closing price on day t; D_t: the dividend paid in $(t-1, t]$
  - k: the number of periods aggregated, indexed by $j = 0, \ldots, k-1$
  - $R_{i,t}$: the simple return of asset i; w_i: its weight in portfolio value, with $\sum_i w_i = 1$; $R_{p,t}$: the portfolio return
- With dividends use total-return or adjusted series (BET-TR)

▸ Further Topics: log below simple

From Day to Year: the Annual Mean and the $\sqrt{A}$ Rule

- A: the number of return intervals per year
  - 252 for stocks and EUR/RON (trading days), 365 for crypto (traded every day), ≈ 260 for gold (the actual weekday quotes)
- $\mu_{\text{ann}} = A\,\bar{r}$, with $\bar{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$: the annual mean of **log** returns
  - the log growth rate
  - needs only a constant mean
  - the Sharpe ratio uses the annual mean of **simple** returns, $A\bar{R}$, which is larger
- $\sigma_{\text{ann}} = \sqrt{A}\,\hat{\sigma}$, with $\hat{\sigma}$ the daily sample standard deviation
  - exact only for serially uncorrelated returns with constant variance
  - then daily variances add up, and the annual variance is $A\,\hat{\sigma}^2$
  - with autocorrelation the $\sqrt{A}$ factor is no longer exact: if the EUR/RON $\hat{\rho}_1 = 0.17$ were the parameter of a first-order autoregressive model, AR(1), the annualised volatility would be about 5.3% instead of 4.4% (▸ Appendix)
  - the empirical scaling needs the whole autocovariance structure: for EUR/RON, negative autocorrelations at other lags offset lag 1 (Chapter 0; Seminar 1, A7)

CAGR and Volatility Drag: Why the Arithmetic Mean Overstates Growth

- **CAGR** (Compound Annual Growth Rate): the constant annual return that takes the price from P_0 to P_T

$$\text{CAGR} = (P_T/P_0)^{1/Y} - 1$$

  - P_0, P_T: the price at the start and at the end of the sample; Y: the number of years
  - CAGR is the annual geometric mean of the growth factors

- On the log scale, the compound growth rate is the arithmetic mean minus half the variance:

$$\ln(1 + \text{CAGR}) = \tfrac{1}{Y} \ln(P_T/P_0) \approx \mu_{\text{arith}} - \sigma^2/2$$

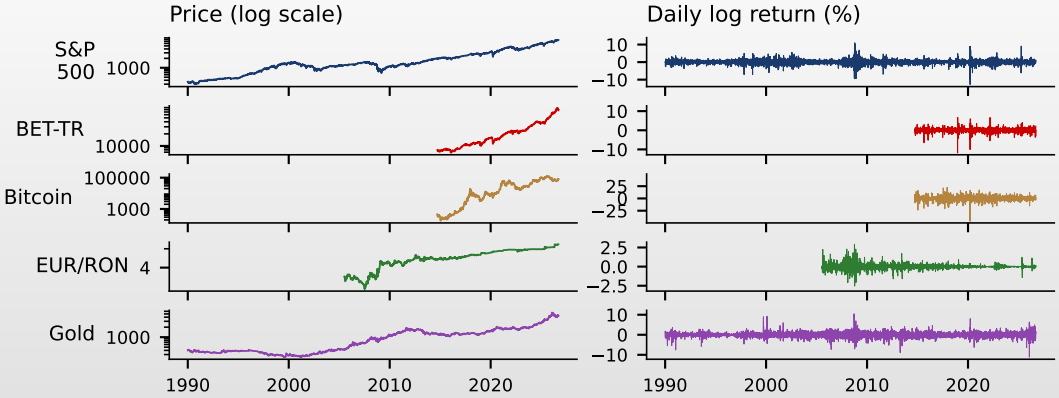
  - μ_{arith}: the annual arithmetic mean of simple returns; σ^2: their annual variance
  - $\ln(1 + \text{CAGR})$: the continuously compounded growth rate, i.e. the annual mean log return $A\bar{r}$
  - **volatility drag**: the gap between the arithmetic and the geometric mean; on the log scale it is $\approx \sigma^2/2$ and grows with the square of volatility

Volatility Drag in Numbers: S&P 500 and Bitcoin

	μ_{arith}	σ	$\sigma^2/2$	$\ln(1+\text{CAGR})$	CAGR
S&P 500 1990–2026 ($A = 252$)	9.95%	17.97%	1.61 pp	8.33%	8.69%
Bitcoin 2014–2026 ($A = 365$)	65.2%	66.2%	21.9 pp	43.1%	53.9%

- Log-scale drag, $\mu_{\text{arith}} - \ln(1 + \text{CAGR})$, is close to $\sigma^2/2$
  - 1.62 pp (percentage points) for the S&P 500
  - 22.1 pp for Bitcoin
- Drag against CAGR, $\mu_{\text{arith}} - \text{CAGR}$: 1.26 pp and 11 pp
  - smaller, because $\text{CAGR} = e^{\ln(1+\text{CAGR})} - 1$ lies above its log: $e^{0.431} - 1 = 53.9\%$
  - when a number is called "drag", say which of the two gaps it is
- **Check**: gold has CAGR 6.7% and $\sigma = 16.3\%$. Estimate μ_{arith}
  - answer: $\ln 1.067 + 0.163^2/2 = 6.5\% + 1.3\% \approx 7.8\%$; with $r_f = 0$ the Sharpe ratio is $\mu_{\text{arith}}/\sigma$, so $0.48 \times 16.3\% \approx 7.8\%$ (appendix table)

Five Markets: Prices and Daily Log Returns



Prices Trend, Returns Fluctuate around a Small Mean

- **Left:** prices on log scale, trending and non-stationary
- **Right:** daily log returns fluctuate around a small mean
  - calm and turbulent periods alternate
  - extremes cluster (2008, March 2020)
- EUR/RON moves more than 1% on only 1.4% of days
- Bitcoin, 12 March 2020: log return -46.5%
  - the price is multiplied by e^{r_t}, so it falls by $1 - e^{-0.465} = 37.2\%$

MFM_ch1_returns

Fact 2, Heavy Tails: the Question and the Tools

- **The question**: are extreme days more frequent than under the Normal distribution?
  - the reference Normal distribution has the same mean and variance
- **Kurtosis** $\hat{K}$ (next slide) and the Jarque–Bera (JB) normality test
  - $\hat{K} > 3$ points to tails heavier than the Normal distribution's
  - inference about kurtosis (the standard error of $\hat{K}$) needs $\mathbb{E}r^8 < \infty$; with a tail index $\alpha \approx 3$ (below) it becomes unreliable
  - the JB (Jarque–Bera, test) test of the null "i.i.d., Normal distribution" stays calibrated: under H_0 all moments exist
  - heavy tails of the alternative lead to rejection, without invalidating the critical values under H_0
  - for normality of the marginal distribution, allowing dependence over time, the calibration must be adapted (Bai & Ng (2005))
- **QQ plot** (quantile–quantile plot): the points $\left(\Phi^{-1}(p_i),\ z_{(i)}\right)$, with $p_i \approx (i - \frac{1}{2})/T$, $i = 1, \ldots, T$
  - $z_{(i)}$: the i-th smallest standardised return; Φ^{-1}: the quantile function of the standard Normal distribution
  - under the Normal distribution the points lie on the line $y = x$; heavy tails bend them away from it at both ends
- The **tail index** α, estimated by the Hill method from the largest values of $|r_t|$
  - needs no finite moments (definition)

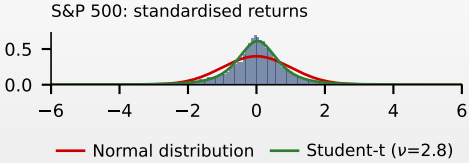
Skewness and Kurtosis: the Shape of the Distribution after Standardisation

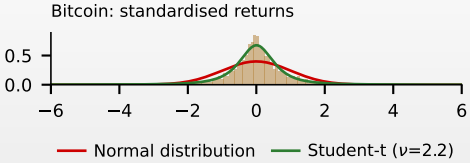
- Sample skewness $\hat{S}$ and sample kurtosis $\hat{K}$ of the standardised return z_t

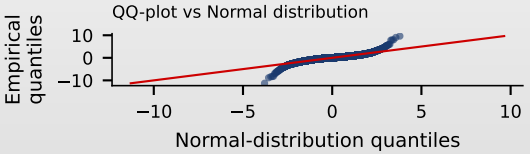
$$z_t = \frac{r_t - \bar{r}}{\hat{\sigma}}, \qquad \hat{S} = \frac{1}{T}\sum_{t=1}^{T} z_t^3, \qquad \hat{K} = \frac{1}{T}\sum_{t=1}^{T} z_t^4$$

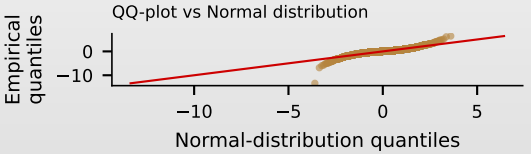
  - $\hat{S} < 0$: the left tail is longer, large losses dominate
  - $\hat{K} > 3$: more mass in the centre and in the tails than under the Normal distribution
- Normal distribution: $S = 0$, $K = 3$, excess kurtosis $K - 3 = 0$

Fact 2, Heavy Tails: Histograms and Quantile–Quantile (QQ) Plots









MFM_ch1_heavy_tails ▶ Interactive

Fact 2: More Extreme Days than under the Normal Distribution

- Histograms: standardised returns have a higher peak and fatter tails than the Normal distribution
  - the reference Normal distribution has the same variance
- QQ-plots: the points bend away from the line $y = x$ in both tails
- A Student-t fits the body of the distribution much better than the Normal distribution
  - whether it also fits the tails is a separate question, answered with the Hill estimator below

MFM_ch1_heavy_tails

What the Tests Need: ARCH, Volatility Clustering, Tail Index

- **ARCH** (AutoRegressive Conditional Heteroskedasticity): today's variance depends on past returns
- **Volatility clustering**: days with large moves tend to follow days with large moves
- The **tail index** α measures how fast the probability of extreme values decays

$$P(|r_t| > x) \approx c\, x^{-\alpha} \quad \text{for large } x$$

  - x: a positive threshold; $c > 0$: a scale constant
  - a small α means a heavy tail: the probability decays slowly as x grows
  - so $\mathbb{E}|r_t|^p < \infty$ only for $p < \alpha$, where p is the order of the moment
  - the estimates in this chapter give $\alpha \approx 3$

Three Classical Tests: Normality, Autocorrelation, ARCH Effects

- **JB** (Jarque–Bera) tests normality: $JB = T\left[\hat{S}^2/6 + (\hat{K}-3)^2/24\right]$
- **LB** (Ljung–Box) tests autocorrelation over the first $m = 10$ lags: $Q(m) = T(T+2)\sum_{k=1}^{m} \hat{\rho}_k^2/(T-k)$
  - $\hat{\rho}_k$: the sample autocorrelation between r_t and r_{t-k}
- **ARCH-LM** (ARCH Lagrange Multiplier, test) tests whether squared returns are predictable from the past
  - statistic: TR^2 from the regression of $\hat{\varepsilon}_t^2$ on $\hat{\varepsilon}_{t-1}^2, \ldots, \hat{\varepsilon}_{t-q}^2$
  - LM: the Lagrange multiplier test; $\hat{\varepsilon}_t = r_t - \bar{r}$: the residuals; $q = 5$ lags
- Reference distributions under H_0: χ_2^2, χ_m^2, χ_q^2
  - large values reject H_0
  - 5% critical values: $\chi_2^2 = 5.99$, $\chi_5^2 = 11.1$, $\chi_{10}^2 = 18.3$

Further Topics: JB limits, bootstrap

Summary Statistics: Five Markets

	S&P 500	BET-TR	Bitcoin	EUR/RON	Gold
Observations	9245	2992	4384	5352	9559
Mean (% per day)	0.033	0.077	0.118	0.007	0.025
Std. dev. (% per day)	1.13	0.98	3.49	0.28	1.01
Annualised volatility (%)	18.0	15.5	66.6	4.4	16.3
Skewness	−0.36	−1.40	−0.70	0.73	−0.23
Excess kurtosis	10.9	17.8	12.0	15.1	9.3
Min / max log return (%)	−12.8 / 11.0	−11.9 / 6.7	−46.5 / 22.5	−2.5 / 2.9	−11.1 / 10.4
Jarque–Bera	45805	40433	26482	51575	34732
Ljung–Box $Q(10)$, r_t	91.6	70.0	20.2	221.7	32.1
Ljung–Box $Q(10)$, r_t^2	8018	714	213	2420	1049
ARCH-LM (5 lags)	2229	371	115	871	483

- Descriptive: heavy tails everywhere (excess kurtosis 9.3–17.8)
  - $Q(10)$ on r_t^2 is 10–90 times larger than on r_t
  - the size of returns is far more predictable than their direction
- Inferential: all i.i.d. p-values are below 0.03
  - but they are valid only for i.i.d. returns with the required moments
  - Ljung–Box on r_t^2 and ARCH-LM require at least a finite fourth moment, which the Hill estimates question

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▸ Further Topics: the Student-t distribution

The Hill Estimator: the Tail Index from the Largest Observations

- The tail index α of $X = |r_t|$
  - $\alpha > 2$: finite variance
  - $\alpha > 4$: finite kurtosis
  - power laws were first argued by Mandelbrot (1963) and Fama (1965)
- The **Hill estimator**, Hill (1975): estimates α from the k largest observations only
  - the descending order statistics $X_{(1)} \geq X_{(2)} \geq \cdots \geq X_{(n)}$ of the n observations
  - k: the number of tail observations; $X_{(k+1)}$ is the threshold where the tail starts
- The **Pareto tail**, $P(X > x) = (x/u)^{-\alpha}$ for $x \geq u$, is the exact power law
  - u is the threshold

The Hill Estimator: the Inverse Mean Log Exceedance

- The Hill estimator is the inverse of the mean log exceedance over the threshold

$$\hat{\alpha}_k = \left[\frac{1}{k} \sum_{i=1}^{k} \ln X_{(i)} - \ln X_{(k+1)} \right]^{-1} = \left[\frac{1}{k} \sum_{i=1}^{k} \ln \frac{X_{(i)}}{X_{(k+1)}} \right]^{-1}$$

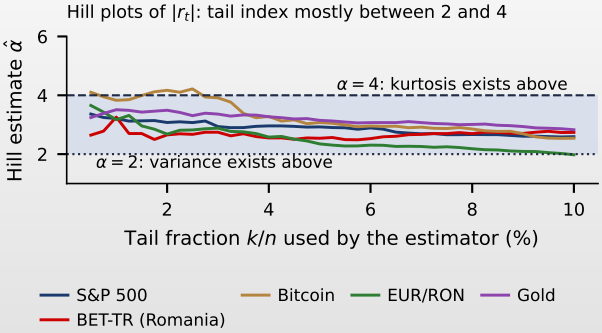
  - $\ln(X_{(i)}/X_{(k+1)})$: how far the i-th value exceeds the threshold, on a log scale
  - for a Pareto tail these exceedances average $1/\alpha$
  - large exceedances mean a heavy tail, hence a small $\hat{\alpha}_k$

- Standard error $\mathrm{SE}(\hat{\alpha}_k) \approx \hat{\alpha}_k/\sqrt{k}$; 95% confidence interval (CI): $\hat{\alpha}_k \pm 1.96\,\mathrm{SE}$
  - it assumes independent tail observations and a Pareto tail
  - clustering of extremes makes it larger (Appendix)

The Hill Estimator by Hand: a Five-Number Example

- Toy data: the five largest $|r_t|$ of a sample, in %: 4.0, 3.0, 2.5, 2.2, 2.0
  - take $k = 4$, so the threshold is $X_{(5)} = 2.0$
- Log exceedances $\ln(X_{(i)}/X_{(5)})$, $i = 1, \ldots, 4$
  - $\ln 2 = 0.693$, $\ln 1.5 = 0.405$, $\ln 1.25 = 0.223$, $\ln 1.1 = 0.095$
  - mean $1.417/4 = 0.354$, so $\hat{\alpha}_4 = 1/0.354 = 2.82$
- SE $\approx 2.82/\sqrt{4} = 1.41$
  - 95% CI $[0.06, 5.59]$
  - with 4 tail points the estimate says almost nothing; this is why the tables use k in the hundreds
- **Check**: double every value (a market twice as volatile). What happens to $\hat{\alpha}_4$?
  - answer: nothing; the ratios $X_{(i)}/X_{(5)}$ are unchanged, so a change of scale does not move the tail index
- Full worked exercise on market data: Seminar 1, A5

Hill Plots for Five Markets



- $\hat{\alpha}$ for $|r_t|$ at $k = 2.5\%$ of n
  - S&P 500 3.1, BET-TR 2.7, Bitcoin 4.2, EUR/RON 2.8, gold 3.3
- For $k = 1$–5%: between 2.3 and 3.5 except Bitcoin (3.1–4.2)
- Tails are modelled with extreme value theory (Chapter 7)

MFM_ch1_heavy_tails ▶ Interactive

Tail Indices with Confidence Intervals

$\lvert r_t \rvert$, $k = 2.5\%$ of n	n	k	$\hat{\alpha}$	SE i.i.d.	95% CI (i.i.d.)	SE block bootstrap
S&P 500	9245	231	3.07	0.20	[2.68, 3.47]	0.33
BET-TR	2992	74	2.67	0.31	[2.06, 3.28]	0.45
Bitcoin	4384	109	4.22	0.40	[3.43, 5.01]	0.47
EUR/RON	5352	133	2.82	0.24	[2.34, 3.30]	0.28
Gold	9559	238	3.31	0.21	[2.89, 3.73]	0.24

- Moving-block bootstrap keeps the clustering of extremes
  - blocks of 20 days, 500 resamples
  - SEs are 1.1–1.6 times the i.i.d. ones

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The Estimates Cast Doubt on a Finite Fourth Moment

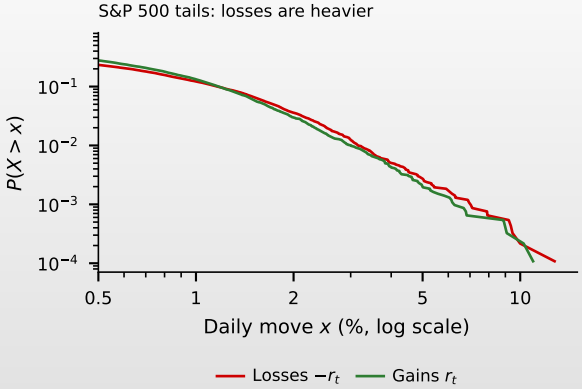
- $\alpha = 4$ is outside every interval except Bitcoin's
  - also when the interval uses the bootstrap SE (Standard Error)
  - the estimates cast doubt on the existence of the fourth moment
  - this is conditional on the Pareto tail approximation and on the threshold used ($k = 2.5\%$ of n)
  - consequence: kurtosis and the tests on r_t^2 have no standard inference (table in Further Topics)
- $\alpha = 2$ is outside every i.i.d. interval
  - with the bootstrap SE only BET-TR reaches it: the lower limit $\hat{\alpha} - 1.96\,\mathrm{SE} = 2.67 - 1.96 \times 0.45 = 1.79$
- Bitcoin's 4.2 uses a threshold where its Hill plot is not yet flat
  - at a common threshold the loss tail gives 2.71 and the gain tail 3.36 (Appendix)

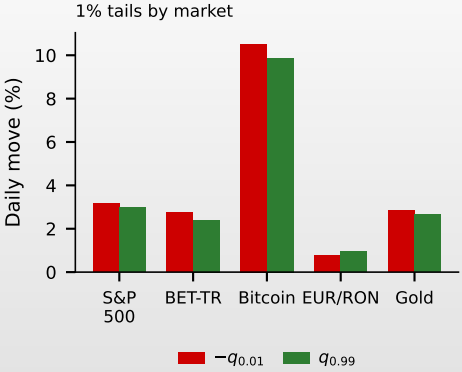
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Fact 3: How We Measure and Test the Gain/Loss Asymmetry

- **The question**: are large losses more frequent than gains of the same size?
- The empirical **survival function** $\hat{P}(X > x) = \frac{1}{T} \#\{t : X_t > x\}$
  - $X_t = -r_t$ for losses and $X_t = r_t$ for gains
  - on a log–log scale a Pareto tail $P(X > x) = (x/u)^{-\alpha}$ is a straight line with slope $-\alpha$
- **The statistic** $\Delta = -q_{0.01} - q_{0.99}$: $\Delta > 0$ means the 1% loss is larger than the 1% gain
  - $-q_{0.01}$ is the 1% **VaR** (Value-at-Risk), the daily loss threshold exceeded with probability 1% (Chapter 7)
- **The test**: $H_0 : \Delta = 0$ is rejected if the 95% moving-block bootstrap CI excludes 0
  - blocks of 20 days
  - the blocks keep the clustering of extreme days
- For the far tails: the Hill tail indices α_- (losses) and α_+ (gains), at the same threshold u

Fact 3: Gain/Loss Asymmetry in the Data





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Fact 3: Extreme Losses Exceed Gains, but the Gap Is Rarely Significant

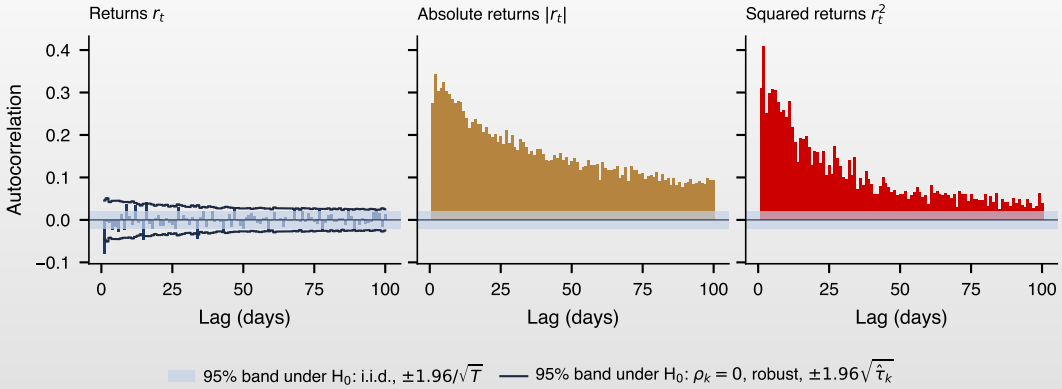
- Left: the S&P 500 survival functions on a log–log scale
  - the loss curve lies above the gain curve
- Right: $-q_{0.01}$ (1% VaR, red) and $q_{0.99}$ (green) in the five markets
- $\Delta > 0$ in 4 of 5 markets, but the 95% CI contains 0 in all four
  - S&P 500: $\Delta = 3.15 - 2.97 = 0.18$ pp, CI $[-0.12,\ 0.49]$ pp
- EUR/RON: $\Delta = -0.17$ pp, CI $[-0.35,\ -0.07]$ pp
  - leu depreciations (up-moves) are significantly larger
- At a common threshold the loss tail is heavier for BET-TR and, marginally, for Bitcoin
  - BET-TR: $\hat{\alpha}_- = 2.20$ versus $\hat{\alpha}_+ = 3.06$, $z = -2.7$ with block-bootstrap SEs
  - Bitcoin: 2.71 versus 3.36, $z = -1.9$
- A Normal-distribution model with the right variance underestimates both extreme quantiles

MFM_ch1_heavy_tails

Facts 1 and 6: the Questions, the Statistics and the Bands

- **Fact 1**: does the direction of returns repeat?
  - statistic: $\hat{\rho}_k(r)$, the autocorrelation between r_t and r_{t-k} at lag k (days); as a function of k: the autocorrelation function (ACF)
- **Fact 6**: does their size repeat (volatility clustering)?
  - statistics: $\hat{\rho}_k(|r|)$ and $\hat{\rho}_k(r^2)$
  - r_t^2 links directly to the variance, as in ARCH and GARCH (generalised ARCH) models
  - $|r_t|$ needs fewer moments: the ACF (AutoCorrelation Function) of $|r_t|$ has a Normal limit for $\alpha > 4$, that of r_t^2 only for $\alpha > 8$ Davis & Mikosch (1998)
- **The test** at each lag: $H_0 : \rho_k = 0$, rejected when $\hat{\rho}_k$ leaves a 95% band
  - the band is centred at 0, the value under H_0: it is a rejection band, not a confidence interval (which would be centred at $\hat{\rho}_k$)
- Two bands: i.i.d., $\pm 1.96/\sqrt{T}$, and robust to volatility clustering, $\pm 1.96\sqrt{\hat{\tau}_k}$ (Further Topics)

Facts 1 and 6: Uncorrelated but Not Independent



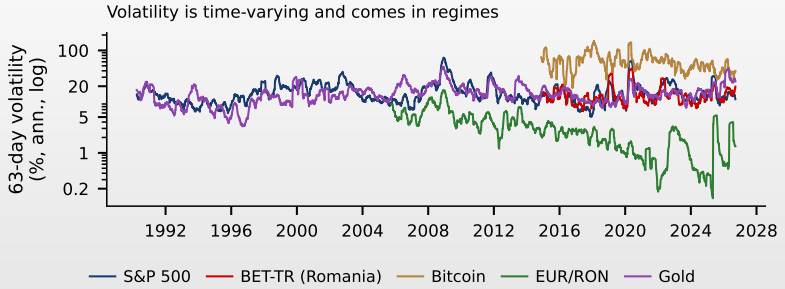
MFM_ch1_volatility_clustering ▶ Interactive

Facts 1 and 6: Direction Does Not Repeat, Size Does

- Left, r_t (S&P 500, 1990–2026): $\hat{\rho}_1 = -0.08$
  - 20 of 100 lags leave the i.i.d. band (blue area), but only 7 leave the robust band (lines)
  - at 5%, about 5 lags in 100 leave the band by chance: apart from lag 1, Fact 1 holds
- Middle and right: $\hat{\rho}_k(|r|)$ and $\hat{\rho}_k(r^2)$ are positive at all 100 lags: Fact 6
  - $\hat{\rho}_1(|r|) = 0.27$
  - the bands require a finite fourth moment, which the Hill estimates question
- Autocorrelations of returns are generally small, with exceptions (S&P 500, EUR/RON), but returns are not independent
  - independence requires zero correlation of **every** function of returns
- The model $r_t = \sigma_t \varepsilon_t$ (GARCH, Chapter 5)
  - σ_t: the conditional volatility on day t, driven by past returns; ε_t: i.i.d. shocks with mean 0 and variance 1

MFM_ch1_volatility_clustering

Fact 5: Intermittency in the Data



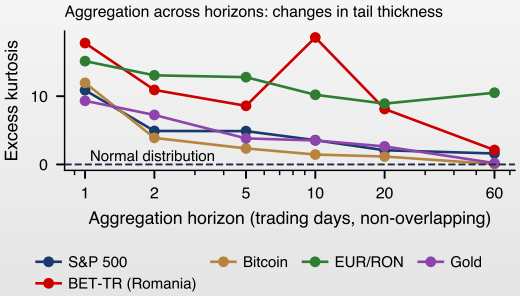
- Bursts of high volatility separated by calm periods
  - constant-volatility models cannot produce them
- 63-day rolling volatility $\sqrt{A}\times$ the standard deviation over the window, log scale
  - common spikes: 2008–2009, March 2020
- EUR/RON volatility collapsed after June 2013, under the BNR's managed float

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Fact 4: Aggregational Gaussianity, Statistic and Test

- **The question**: do h-day returns, $r_t(h) = r_t + \cdots + r_{t-h+1}$, approach the Normal distribution as h grows?
- Statistic: the excess kurtosis $\hat{K}(h) - 3$ of non-overlapping block sums
  - n_h: the number of blocks
- Test: $z = (\hat{K}(h) - 3)/\sqrt{24/n_h} \approx N(0,1)$ under H_0
  - H_0: $r_t(h)$ i.i.d. Normal
  - under H_0 all moments exist

Fact 4: Aggregational Gaussianity



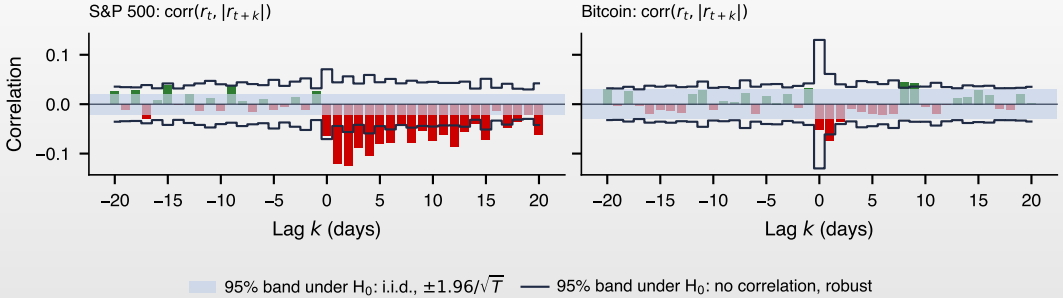
- The chart: $\hat{K}(h) - 3$ against h in trading days
  - dotted line: the Normal distribution
- At $h = 60$ (a quarter for the S&P 500, two months for Bitcoin)
  - S&P 500: $10.9 \to 1.6$ ($n_h = 154$, $z = 4.1$)
  - Bitcoin: $12.0 \to 0.1$ ($n_h = 73$, $z = 0.1$)
- Normality is still rejected at $h = 60$ for the S&P 500, BET-TR and EUR/RON, but not for Bitcoin and gold
  - BET-TR: $z = 3.0$
  - EUR/RON: $15.1 \to 10.5$, $z = 20$
- The CLT (Central Limit Theorem) works slowly here: daily returns are dependent and heavy-tailed

MFM_ch1_heavy_tails ▶ Interactive

Fact 9: the Leverage Effect, Statistic and Test

- **The question**: does volatility rise more after falls than after rises?
- Statistic: $L(k) = \text{corr}(r_t, |r_{t+k}|)$
  - the correlation of today's return with the size of the return k days later
  - $L(k) < 0$ for $k > 0$: after falls $|r_t|$ grows, i.e. volatility rises
  - $k < 0$ is a control: the size of past returns should not predict the sign of today's
- **The test**: $H_0 : L(k) = 0$, with $z = \hat{L}(k)/\text{SE}$ compared with ± 1.96
  - i.i.d. SE $= 1/\sqrt{T}$: ignores volatility clustering, so the $\pm 1.96/\sqrt{T}$ band is too narrow
  - robust HAC (heteroskedasticity and autocorrelation consistent) SE $= \sqrt{\hat{\tau}}$: $\hat{\tau}$ is the Newey–West long-run variance (Bartlett, $\ell = \lfloor 4(T/100)^{2/9} \rfloor$ lags) of $u_t = x_t y_{t+k}/(s_x s_y)$, divided by T
  - $x_t = r_t - \bar{r}$, $y_t = |r_t| - \overline{|r|}$; s_x, s_y: their standard deviations
  - assumes weak dependence of u_t: even with $L(k) = 0$ the products u_t can be autocorrelated, and their autocorrelations up to lag ℓ enter the SE

Fact 9: The Leverage Effect



MFM_ch1_leverage_volume ▶ Interactive

Fact 9: Volatility Rises after Falls

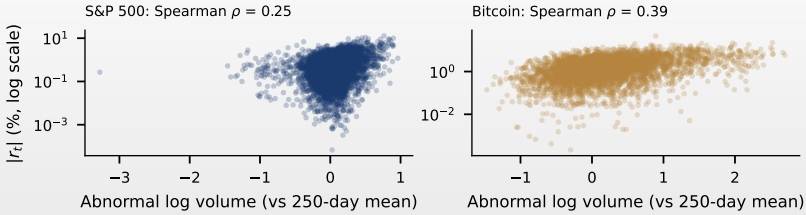
- The chart: $\hat{L}(k)$ for k from -20 to 20
  - blue area: the i.i.d. band
  - lines: the robust band
- S&P 500: $L(1) = -0.12$ ($z = -5.4$, robust SE)
  - the robust SE requires a finite fourth moment, which the Hill estimates question
  - still -0.07 at $k = 10$
  - for $k < 0$ no value is significant
- $L(1)$ for BET-TR, Bitcoin and gold
  - BET-TR: $L(1) = -0.12$ ($z = -2.4$)
  - Bitcoin: -0.07 ($z = -2.4$)
  - gold: -0.02 ($z = -0.8$; we do not reject $L(1) = 0$)
- EUR/RON: reversed, $L(1) = +0.10$ ($z = 2.5$)
  - volatility rises after leu depreciations
- Type of evidence: association (the variables move together; this does not prove causation)
  - possible explanations: the financial leverage of firms Christie (1982), the retarded volatility model Bouchaud, Matacz & Potters (2001)
- Asymmetric GARCH models (Chapter 5): GJR-GARCH (Glosten–Jagannathan–Runkle), EGARCH (exponential GARCH)

MFM_ch1_leverage_volume

Fact 10: Volume and Volatility, Statistic and Test

- **The question**: are days of unusually high volume also days of large price moves?
- Abnormal volume: $\tilde{v}_t = \ln V_t - \frac{1}{250}\sum_{j=0}^{249} \ln V_{t-j}$
  - V_t: the volume traded on day t; the moving average removes the long-run trend in volume
- Statistic: the Spearman rank correlation $\rho_S = \mathrm{corr}\big(\mathrm{rank}(\tilde{v}_t),\ \mathrm{rank}(|r_t|)\big) \in [-1, 1]$
  - ranks make it robust to the heavy tails of both series and to monotone non-linear relations
- **The test**: $H_0 : \rho_S = 0$ is rejected if the 95% moving-block bootstrap CI excludes 0
  - blocks of 20 days
  - the blocks keep the persistence of both series

Fact 10: Volume and Volatility Move Together



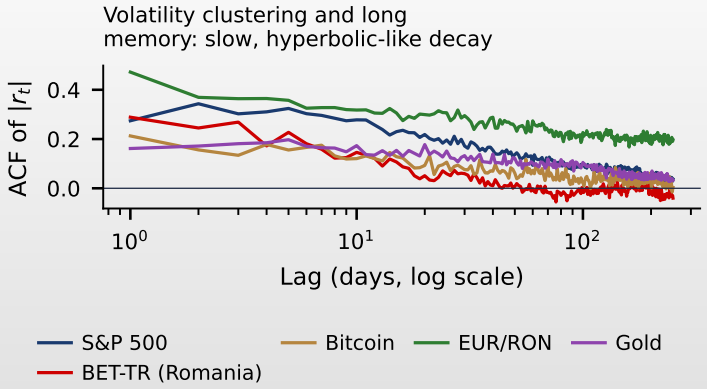
- Axes: abnormal volume $\tilde{v}_t$ (horizontal) and $|r_t|$ on a log scale (vertical); one point is one day
- $\hat{\rho}_S = 0.25$ for the S&P 500 and 0.39 for Bitcoin: a positive, significant relation
  - CI: [0.22, 0.28] for the S&P 500, [0.34, 0.44] for Bitcoin
- Type of evidence: association
- Mixture-of-distributions hypothesis Clark (1973)
  - information flow drives both price moves and volume (survey of the evidence: Karpoff (1987))
- Bitcoin volume is exchange-reported and partly **wash trading** Cong, Li, Tang & Yang (2023)
  - wash trading: fictitious trades in which buyer and seller coincide

MFM_ch1_leverage_volume ▶ Interactive

Fact 8: Exponential or Hyperbolic Decay?

- **The question**: how fast does ρ_k, the autocorrelation of $|r_t|$ at lag k, decay as k grows?
- **Short memory**: $\sum_k |\rho_k| < \infty$
  - e.g. exponential decay $\rho_k \approx c\,\phi^k$, with $c > 0$ and $0 < \phi < 1$ (GARCH(1,1), Chapter 5)
- **Long memory** Ding, Granger & Engle (1993): hyperbolic decay $\rho_k \approx c\,k^{2d-1}$, with $0 < d < \frac{1}{2}$
  - d: the memory parameter; a larger d means slower decay, and $\sum_k \rho_k = \infty$
- **The test**: estimate d and test $H_0 : d = 0$ (short memory)
  - with the semiparametric estimators in Further Topics

Fact 8: Slow Decay of the ACF of $|r_t|$



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Fact 8: the ACF of $|r_t|$ Decays Slowly in Every Market

- The chart: the ACF of $|r_t|$ (vertical) against the lag (horizontal, log scale), lags 1–250
- S&P 500: 0.27 at lag 1, 0.22 at lag 20, still 0.09 at lag 100
- BET-TR decays fastest
- Slow empirical decay is not yet established long memory
  - structural breaks, i.e. abrupt shifts in the level of volatility, can imitate it

MFM_ch1_volatility_clustering

$\hat{d} \approx 0.35$–0.50 in All Five Markets

$\lvert r_t \rvert$, $m = T^{0.65}$	S&P 500	BET-TR	Bitcoin	EUR/RON	Gold
m	378	181	232	265	386
Local Whittle $\hat{d}$ (SE)	0.48 (0.03)	0.37 (0.04)	0.35 (0.03)	0.39 (0.03)	0.40 (0.03)
Log-periodogram $\hat{d}$ (SE)	0.50 (0.03)	0.41 (0.05)	0.35 (0.04)	0.38 (0.04)	0.42 (0.03)

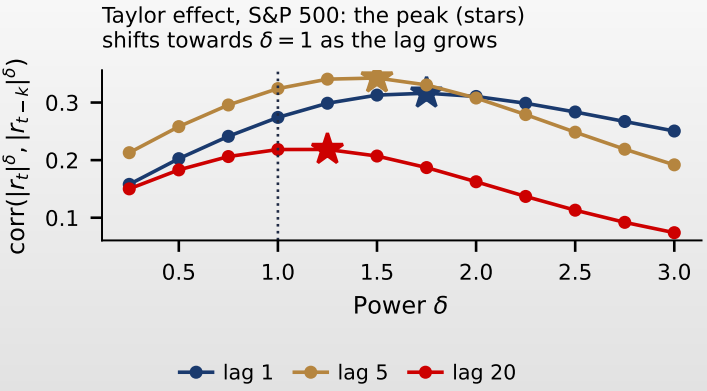
- Every $\hat{d}$ is at least 8 SEs from $d = 0$ (short memory), with both estimators
- The S&P 500 is close to the stationarity boundary: for $d \geq \frac{1}{2}$ the series is no longer stationary

MFM_ch1_inference

The Taylor Effect: the Question and the Statistic

- **The question**: which power of the size of returns is the most persistent?
- Statistic: $\rho_k(\delta) = \text{corr}(|r_t|^\delta, |r_{t-k}|^\delta)$, as a function of the power $\delta > 0$
  - $\delta = 1$: absolute returns; $\delta = 2$: squared returns
- **Taylor effect**: $\rho_k(\delta)$ peaks at a power δ close to 1, not at $\delta = 2$
  - documented by Taylor (1986), named by Granger & Ding (1995)
- The effect motivates power-GARCH models (Chapter 5)
  - power-GARCH models σ_t^δ with an estimated δ
  - it also motivates absolute-value volatility measures

The Taylor Effect



MFM_ch1_volatility_clustering ▶ Interactive

The Taylor Effect: Persistence Peaks below the Square

- The chart: $\rho_k(\delta)$ for lags 1, 5 and 20
  - stars mark the peak
  - dotted line: $\delta = 1$
- S&P 500: maximum at $\delta = 1.75$ (lag 1), 1.5 (lag 5) and 1.25 (lag 20)
  - at the three lags in the chart the peak is below 2 and moves towards 1 as the lag grows; not at every lag (at lag 2 it is exactly 2)
  - at lag 1 the peak is closer to 2 than to 1: here the evidence for "$\delta \approx 1$" is mixed at short lags
- The location of the peak is a point estimate without a standard error
  - the comparison with $\delta = 1$ is descriptive

MFM_ch1_volatility_clustering

Facts 7 and 11: Conditional Tails and Time Scales

Fact 7: conditional heavy tails

- Fit a volatility model, e.g. GARCH (Chapter 5)
  - and standardise: $\hat{z}_t = r_t / \hat{\sigma}_t$, with $\hat{\sigma}_t$ the model's volatility estimate for day t
- The kurtosis of $\hat{z}_t$ is lower than that of r_t, still above 3 (literature; not tested here)
- Models need heavy-tailed innovations (Student-t, possibly skewed)

Fact 11: asymmetry in time scales

- Weekly volatility predicts daily volatility better than the reverse
- Traders with different horizons react at different frequencies
- The heterogeneous autoregressive model of realised volatility (**HAR-RV**, Chapter 9)
  - realised volatility is the sum of squared intraday returns

- Returns standardised by realised volatility from intraday data are close to the Normal distribution
  - Andersen, Bollerslev, Diebold & Ebens (2001) (Chapter 9)

Differences between the Five Markets

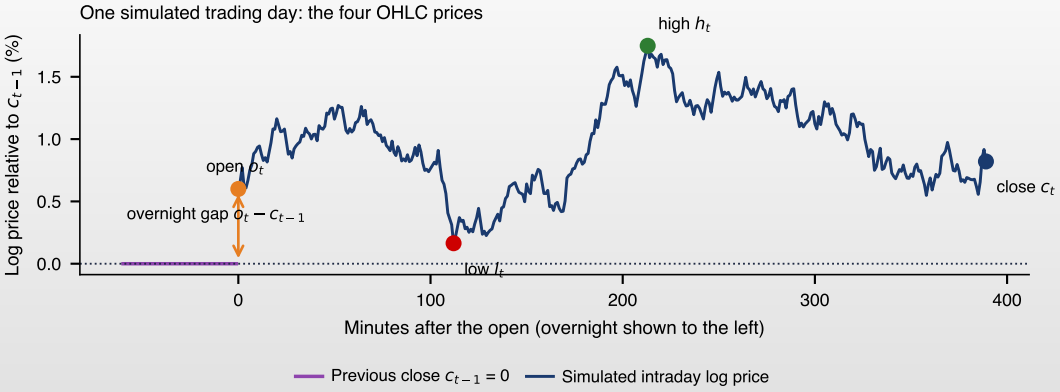
Market	Institutional feature	What differs	Candidate explanation
Bitcoin	trades 24/7; $A = 365$	volatility 66.6% versus 18.0% (S&P 500); worst day -46.5%; weak leverage (-0.07); ACF of $\lvert r_t \rvert$ at lag 100: 0.02 versus 0.09	no corporate leverage; volatility rises after large moves of either sign
EUR/RON	BNR daily fixing, managed float	volatility 4.4% but excess kurtosis 15.1; $\hat{\rho}_1 = +0.17$; largest moves October 2008 ($+2.9\%$, -2.5%)	adjustments smoothed over several days
BET-TR	total return, frontier/emerging market	skewness -1.40; worst day -11.9% on 19 Dec 2018 (bank-asset tax announced on 18 Dec, Government Emergency Ordinance 114/2018); $\hat{\rho}_1 = +0.06$	policy shocks; **non-synchronous trading** of constituents Campbell, Lo & MacKinlay (1997), ch. 3

- The explanations are hypotheses, not identified causes
- Low volatility is not low tail risk
- Heavy tails and volatility clustering hold in all five markets
  - autocorrelation and asymmetry depend on institutions

What the Day's High and Low Add

- **The question**: how precisely can we estimate a day's volatility if we also use the open, the high and the low?
  - not only the close
- **OHLC**: four prices of day t
  - O_t: the open; H_t: the high; L_t: the low; C_t: the close
- The next chart: one simulated day
  - vertical axis: the log price relative to the previous close $c_{t-\mathbf{1}}$, in %
  - the overnight gap $o_t - c_{t-\mathbf{1}}$ sits left of the open
- **Range-based** estimators use the whole path of the day
  - the range H_t/L_t is large on volatile days, even if the close equals the open

Open, High, Low and Close (OHLC) Prices of One Day



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The Log Range, the Day's Returns and the Efficiency of an Estimator

- Log prices: $o_t = \ln O_t$, $h_t = \ln H_t$, $l_t = \ln L_t$, $c_t = \ln C_t$
- Log range of day t: $h_t - l_t = \ln(H_t/L_t) \geq 0$
  - approximately the percentage gap between the day's high and low
- Three log returns of day t
  - close-to-close: $c_t - c_{t-1}$
  - overnight: $o_t - c_{t-1}$
  - intraday: $c_t - o_t$
- **Close-to-close estimator**: the standard deviation of $c_t - c_{t-1}$, one observation per day
- **Efficiency** of an estimator X: $\mathrm{Var}(\hat{\sigma}^2_{CC})/\mathrm{Var}(\hat{\sigma}^2_X)$
  - an efficiency of 5 means the same precision with 5 times fewer days

The Parkinson Estimator: Variance from the Day's Range

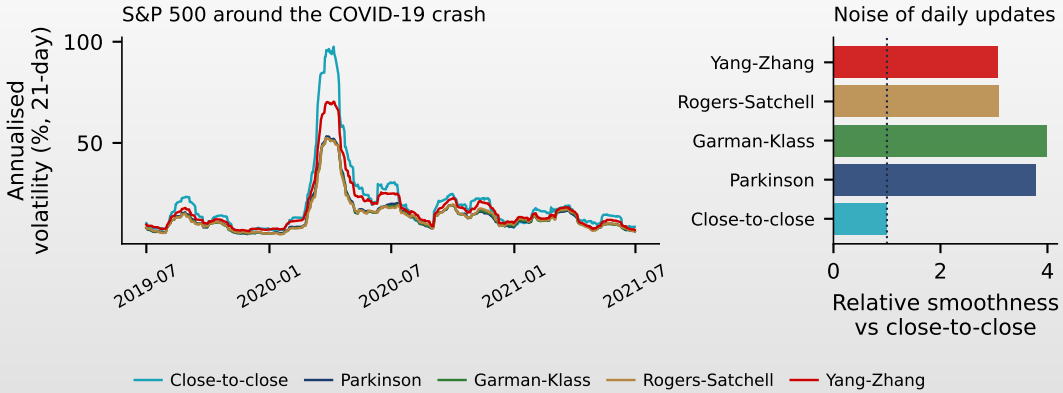
- **Parkinson**, Parkinson (1980): estimates the daily variance from the daily range only

$$\hat{\sigma}_P^2 = \frac{1}{4 \ln 2} \frac{1}{n} \sum_{t=1}^{n} (h_t - l_t)^2$$

  - for a Brownian motion with daily variance σ^2, $\mathbb{E}[(h_t - l_t)^2] = 4 \ln 2 \, \sigma^2$: dividing by $4 \ln 2$ makes the estimator unbiased
  - $\hat{\sigma}_P^2$ is a daily variance (n days); the annual volatility is $\sqrt{A \, \hat{\sigma}_P^2}$

- Assumption: a Brownian motion without drift and without overnight jumps
  - Brownian motion: the continuous-time random walk (Chapter 11)
  - efficiency relative to close-to-close: about 5 (Seminar 1, A8)

Volatility Estimators on the S&P 500



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Missing the Overnight Jumps, Range Estimators Read Lower

- 21-day windows, 2008–2026; averages
  - close-to-close 16.4%, Yang–Zhang 13.9%, Parkinson 13.0%, Garman–Klass 12.4%, Rogers–Satchell 12.3%
- Pure range estimators are lower: they miss overnight gaps
- Right panel, "relative smoothness" of estimator X

$$\left[\frac{\text{std. dev. of daily changes, close-to-close estimate}}{\text{std. dev. of daily changes, estimate } X}\right]^2$$

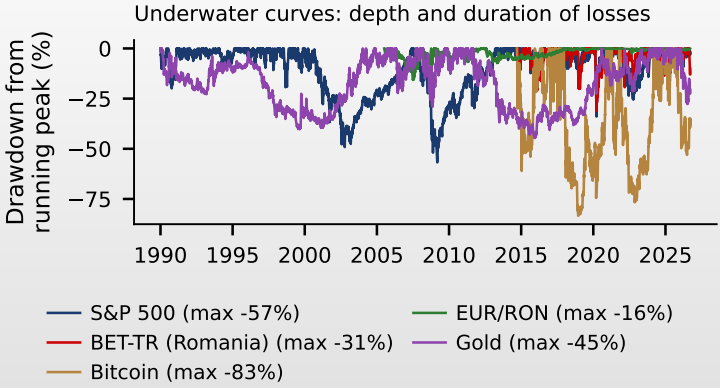
  - equal to 1 for close-to-close; larger values mean the estimate jumps less from one day to the next
- The range-based series are 3–4 times smoother on this measure
  - a property of the estimates, not by itself proof of efficiency: it does not compare them with the true σ

MFM_ch1_vol_estimators

Drawdowns: the Question, the Statistic and the Benchmark

- **The question**: is the deepest observed fall larger than a random walk would produce?
  - a random walk with the same drift and volatility
- Drawdown $DD_t = P_t / \max_{s \le t} P_s - 1 \in [-1, 0]$ (Chapter 0)
  - maximum drawdown: $MDD = \min_t DD_t$
- On the log scale: $D = \max_t \left(\max_{s \le t} \ln P_s - \ln P_t \right) = -\ln(1 + MDD) \ge 0$
- The benchmark Magdon-Ismail, Atiya, Pratap & Abu-Mostafa (2004): $\ln P_t$ is a Gaussian random walk with annual drift μ and annual volatility σ, over Y years
  - $\mu = 0$: $\mathbb{E}[D] = \sqrt{\pi/2}\,\sigma\sqrt{Y}$, growing in proportion to $\sqrt{Y}$
  - $\mu > 0$: $\mathbb{E}[D] \approx \frac{2\sigma^2}{\mu}\left[\frac{1}{4}\ln\frac{\mu^2 Y}{2\sigma^2} + 0.49\right]$, growing only like $\ln Y$ (an approximation for large $\mu^2 Y/(2\sigma^2)$)
- **The test**: p-value $= P(D \ge D_{\text{obs}})$ under the benchmark, from 5,000 paths simulated with the estimated drift and volatility

Drawdowns Across Markets



MFM_ch1_risk_measures ▶ Interactive

The Maximum Drawdown: How Far a Price Can Fall from a Peak

- The chart: underwater curves $100\,DD_t$ (%)
  - the legend gives each market's MDD
- MDD: Bitcoin -83%, S&P 500 -57%, gold -45%, BET-TR -31% (March 2020), EUR/RON -16%
- S&P 500: 9 October 2007 (1,565.15) to 9 March 2009 (676.53): -56.8%
- **Recovery**: the price regaining its previous peak
  - after a fall of x it needs a gain of $1/(1-x)-1$; here $1/(1-0.568)-1=+131\%$

MFM_ch1_risk_measures

The Sharpe Ratio: Excess Return per Unit of Volatility

- **Sharpe ratio** Sharpe (1966), Sharpe (1994)

$$SR = \frac{\mu_{\text{arith}} - r_f}{\sigma_{\text{ann}}}$$

  - r_f: the risk-free rate; $r_f = 0$ here, for comparability
  - $\mu_{\text{arith}} = A\bar{R}$: the annual mean of **simple** returns (not the $A\bar{r}$ of the $\sqrt{A}$ rule), with $\bar{R}$ their daily mean
  - σ_{ann}: the annual volatility of simple returns
- S&P 500 1990–2026: $\mu_{\text{arith}} = 9.95\%$, $\sigma_{\text{ann}} = 17.97\%$, $SR = 0.55$
  - with the log-return mean $A\bar{r} = 8.33\%$ one would get $8.33/17.97 = 0.46$

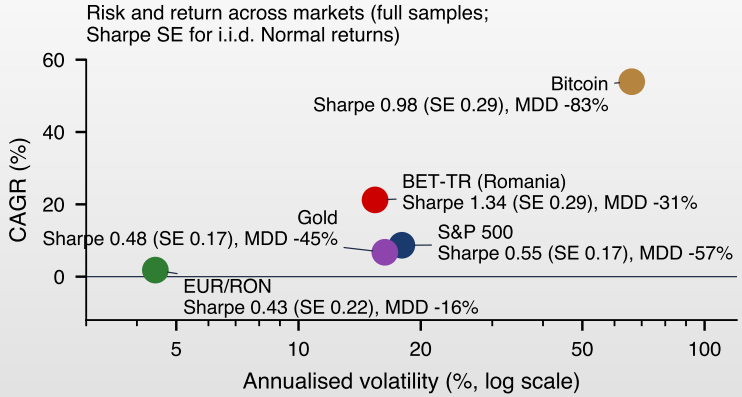
Sharpe Ratios with Standard Errors

- Delta method, i.i.d. returns from the Normal distribution Lo (2002)
  - SR_d: the daily Sharpe ratio, estimated by $\widehat{SR}_d$, with $\mathrm{Var}(\widehat{SR}_d) \approx (1 + SR_d^2/2)/T$
  - annualised, $SR_{\text{ann}} = \sqrt{A}\, SR_d$: $\mathrm{SE}(\widehat{SR}_{\text{ann}}) \approx \sqrt{A(1 + SR_d^2/2)/T} \approx 1/\sqrt{Y}$, with $Y = T/A$ years
  - the span matters, not the frequency
  - the SE requires a finite fourth moment, which the Hill estimates question

	S&P 500	BET-TR	Bitcoin	EUR/RON	Gold
Years Y	36.7	12.0	12.0	21.2	36.7
Sharpe ratio (SE, Normal)	0.55 (0.17)	1.34 (0.29)	0.98 (0.29)	0.43 (0.22)	0.48 (0.17)

MFM_ch1_risk_measures

Risk and Return Across Markets



MFM_ch1_risk_measures ▶ Interactive

Why the Risk–Return Chart Is Not a Ranking

- Axes: annualised volatility (horizontal, log scale) and CAGR (vertical)
- Labels: the Sharpe ratio, its standard error (SE) and the MDD
  - SE: the formula of Sharpe ratios with standard errors, valid for i.i.d. Normal simple returns; full table in the Appendix
- Different sample periods: not a ranking
- With $r_f = 0$ the BET-TR Sharpe ratio is inflated
  - S&P 500 without dividends
  - BET-TR with dividends and in RON (Romanian leu)
  - RON short-term rates were well above US (United States) dollar (USD) rates
- Heavy tails and negative skewness flatter the Sharpe ratio
  - report it with drawdowns, the 1% VaR and the tail index $\hat{\alpha}$

MFM_ch1_risk_measures

Case Study (1/4): How Much of Volatility Do Past Returns Alone Explain?

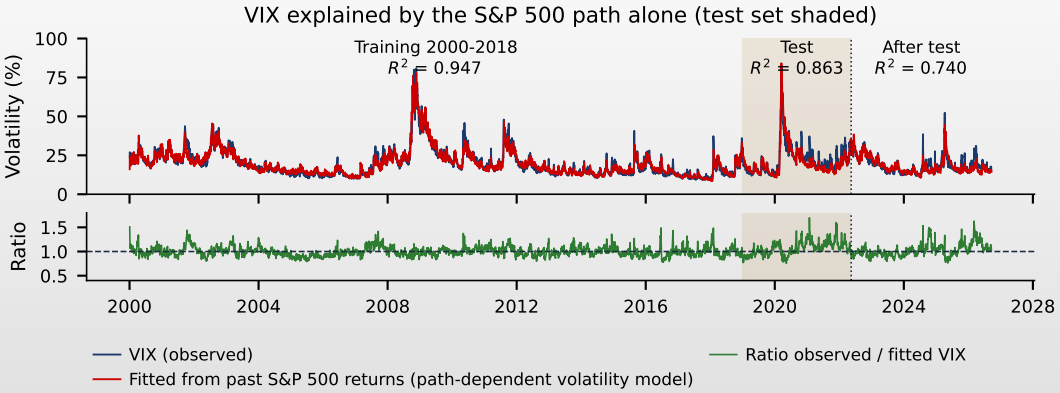
- Paper: Guyon & Lekeufack (2023), *Quantitative Finance*
  - the paper builds on Bollerslev, Hood, Huss & Pedersen (2018)
- Research question
  - what share of an equity index's volatility is explained by its past daily returns alone?
  - what functional form does this dependence take?
- Method: volatility as a linear function of the recent trend and the recent volatility, both weighted averages of past returns (Further Topics)
- Three dependent variables, modelled separately (Sections 3.4–3.5 of the paper)
  - VIX (the Cboe volatility index): the implied volatility of the S&P 500 over the next 30 days, computed from option prices
  - **implied volatility**: the volatility that makes an option-pricing model match observed option prices, i.e. the market's forward-looking volatility (Chapter 12)
  - VIX9D: the same measure over a 9-day horizon
  - next-day realised volatility, computed from 5-minute returns

Case Study (2/4): Past Returns Explain 80–92% of Implied Volatility

Index	Implied volatility (paper, Table 2) R^2 train	R^2 test	Realised volatility (paper, Table 4) R^2 train	R^2 test
S&P 500 (VIX)	0.946	0.855	0.738	0.654
S&P 500 (VIX), our replication	0.947	0.863	–	–
S&P 500 (9-day VIX)	0.876	0.914	–	–
Euro Stoxx 50	0.929	0.913	0.672	0.682
Nikkei 225	0.890	0.800	0.563	0.504

- Past index returns alone explain 80–92% (across the five indices of the paper) of the variance of implied volatility out of sample
  - FTSE 100 (the UK stock index, Financial Times Stock Exchange) and DAX (Deutscher Aktienindex, the German stock index; test R^2 0.918, the maximum) in the Appendix
- On our data (S&P 500 and VIX): RMSE (root mean squared error) 0.020 (2 VIX points) training and 0.034 test
- Realised volatility scores are lower
  - 5-minute realised volatility is a noisy target (Remark 3 in the paper)
- Type of evidence: association
  - past returns predict volatility out of sample; this does not identify what causes it

Case Study (3/4): Observed VIX and VIX Fitted from S&P 500 Returns



MFM_ch1_path_dependent_vol ▶ Interactive

▸ Further Topics: subperiods, kernels

Case Study (4/4): One Kernel for Short and Long Horizons; What Stays Exogenous

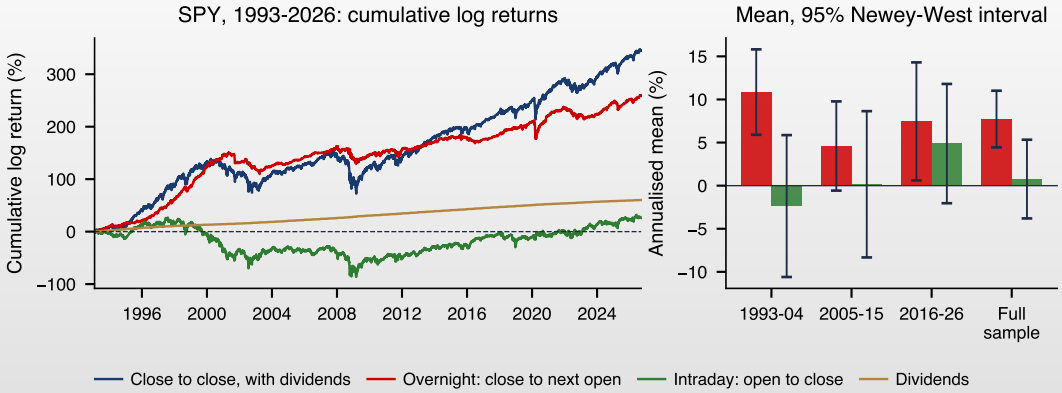
- Daily weights $K(i\Delta t)\Delta t$, log–log scale
  - the kernel K gives the weight of each past day: K_1 for returns (trend), K_2 for their squares (volatility)
  - α: how fast the weights decay; δ: the time shift, in years (not the tail index or the Taylor-effect power)
  - in parentheses: Table 3 of the paper
  - K_1 (trend): $\hat{\alpha}_1 = 1.13$ (1.06), $\hat{\delta}_1 = 0.022$ (0.020)
  - K_2 (volatility): $\hat{\alpha}_2 = 1.58$ (1.60), $\hat{\delta}_2 = 0.050$ (0.052)
- The last 5 days carry 20% of the weight of K_2
  - the days beyond lag 250 still carry 10%
- Limits stated by the authors (Section 5 of the paper)
  - a small exogenous part remains
  - the ratio of observed to fitted VIX mean-reverts around 1
  - this ratio needs a stochastic factor
- Open extension (project seed)
  - does the same formula describe the realised volatility of Bitcoin or of BET-TR?

MFM_ch1_path_dependent_vol

The Market Return Around the Clock (1/4): In Which Hours Is the Mean Return Earned?

- Paper: Bondarenko & Muravyev (2023), *Journal of Financial and Quantitative Analysis*
  - data: E-mini S&P 500 futures, traded almost 24 hours a day
- Research question: does the average market return depend on the time of day?
  - fact 1 (uncorrelated returns) concerns forecasting the return of one day
  - here the question is the mean over a fixed part of the day
- Main result of the paper
  - the 4 hours around the European open account for the entire average market return
  - the Sharpe ratio of these 4 hours is 1.6; their return stays high after transaction costs
  - in the remaining 20 hours, the mean is zero, estimated with much noise
- Explanation proposed by the authors: European investors process the information accumulated overnight
  - the uncertainty in VIX futures prices rises overnight and falls around the European open
- Why it matters: a model with i.i.d. returns within the day spreads the mean evenly over the hours

The Market Return Around the Clock (2/4): SPY Overnight and Intraday



MFM_ch1_overnight_returns ▶ Interactive

The Market Return Around the Clock (3/4): The Paper's Method and the Course Replication

- Paper: returns over intervals of the 24 hours of the futures contract, which pays no dividends
- Course replication: SPY (SPDR S&P 500 ETF Trust), the ETF (Exchange-Traded Fund) on the S&P 500, daily prices, 1 February 1993 – 18 September 2026, 8,466 days
  - SPY, not the index: the open of the S&P 500 index is often stale (Further Topics)
  - overnight $r_t^{\mathbf{on}} = \ln(O_t / C_{t-\mathbf{1}})$; intraday $r_t^{\mathbf{id}} = \ln(C_t / O_t)$; O_t, C_t: opening and closing price
  - dividends $d_t = \ln(f_t / f_{t-\mathbf{1}})$, with f_t = adjusted close / close
  - $r_t^{\mathbf{on}} + r_t^{\mathbf{id}} + d_t$ = the total log return from one close to the next
  - the same decomposition, applied to US (United States) equity strategies: Lou, Polk & Skouras (2019)
- What the replication does differently
  - the "overnight" interval, 16:00 – 9:30 New York time, contains the European open, but also other hours
  - with daily data we cannot isolate the paper's 4 hours
- Annualised mean = 252 × daily mean; t statistic with the Newey–West HAC standard error, as for the leverage effect
- Extension to a 24/7 market: Bitcoin, 5-minute bars, August 2024 – September 2026 (749 days)
  - the return in the US session (9:30 – 16:00 New York, weekdays) against the rest of the day

MFM_ch1_overnight_returns

The Market Return Around the Clock (4/4): The Result and Its Limits

- SPY overnight: mean 7.7% a year ($t = 4.6$), volatility 10.6%, Sharpe ratio 0.73
- SPY intraday: mean 0.8% a year ($t = 0.33$), volatility 15.2%, Sharpe ratio 0.05
  - Sharpe ratios from log returns, with $r_f = 0$; dividends add 1.8% a year
  - 91% of the cumulative log price return was earned overnight
- The overnight mean is positive in all three subperiods: 10.9%, 4.6% and 7.5% a year
  - significant at 5% in 1993–2004 ($t = 4.3$) and 2016–2026 ($t = 2.1$), not in 2005–2015 ($t = 1.7$)
  - the intraday mean is not significant at 5% in any subperiod ($|t| \leq 1.38$)
  - in 23 of 34 calendar years, the overnight return exceeds the intraday return
- Bitcoin: -3.6% a year in the US session, 20.9% in the rest of the day; $|t| \leq 1.0$
  - with two years of data, the difference cannot be told apart from zero
- Type of evidence: a regularity of the mean, stable across subperiods, consistent with the paper
  - not a strategy: the mean is 3.1 basis points (0.031%) per night, and a round-trip cost of 3 basis points cancels it
  - not the cause: the uncertainty-resolution explanation is the authors' and needs intraday futures data
- Project seed: with the 5-minute SPY bars, in which hour of the US session is the mean lost or earned?

MFM_ch1_overnight_returns

From Evidence to Model Requirements

Evidence (five markets)	Model requirement	Where in this course
quantile (Crow–Siddiqui, Further Topics) excess kurtosis > 0 in all markets; Hill $\hat{\alpha} \approx 2.3$–3.5 (Bitcoin: 4.2)	heavy-tailed innovations, also conditionally	Ch. 5, Ch. 7 (extreme value theory)
ACF of $\lvert r_t \rvert$ positive for months; $\hat{d} \approx 0.35$–0.50	persistent, time-varying volatility; long memory or breaks	Ch. 5, Ch. 9
$L(1) = -0.12$ (S&P 500, BET-TR); reversed for EUR/RON	asymmetric response to bad news where it exists	Ch. 5 (GJR-GARCH, EGARCH)
$\hat{\rho}_1$ significant (robust SEs) only for the S&P 500 and EUR/RON; economically small in all markets	approximately uncorrelated returns; market microstructure matters	Ch. 2, Ch. 10
quantile skewness (Further Topics) negative only for the S&P 500	asymmetry is market-specific	Ch. 7–8

- The random walk with i.i.d. increments from the Normal distribution violates Facts 2, 3 and 5–11
  - the Bachelier–Black–Scholes world
  - Facts 1 and 4 hold trivially
- ARCH-LM significance supports volatility clustering descriptively
  - with doubtful fourth moments it is not a formal proof

Open Question: Are Tail Indices Stable over Time?

- The tail index α is usually treated as a constant
- Crypto-assets changed in structure
  - futures (2017), spot Bitcoin ETFs (January 2024), institutional holders
  - Liu & Tsyvinski (2021): crypto returns are not explained by the risk factors of stocks, currencies or precious metals
  - Pele et al. (2023): the tail factor separates cryptos from other asset classes
- Hypothesis proposed by an AI assistant
  - after the launch of the spot ETFs (January 2024), the entry of institutional investors has made the tails of Bitcoin's return distribution thinner
  - i.e. it has raised the tail index α

The Discovery Loop for Bitcoin's Tail Index

- **Literature**: ask for peer-reviewed work with DOIs, then resolve each DOI (Digital Object Identifier) yourself
  - Hoga (2017): change-point tests for the tail index valid under β-mixing dependence
  - β-mixing: dependence that vanishes with the distance in time, e.g. in GARCH models
- **Competing hypotheses**
  - a change in tail shape
  - a fall in volatility only (scale does not move α)
  - sampling noise
- **Data and code**: replicate the chapter's Hill estimates first (Bitcoin 4.2, S&P 500 3.1)
- Fixed in advance
  - $k = 2.5\%$ of n, the break date 11 January 2024
  - block-bootstrap SEs, a placebo asset (the same analysis on an asset for which the hypothesis predicts no change)

Mini-Case: Did Bitcoin's Tail Get Thinner after the ETFs?

- Test: Hill estimate for $|r_t|$, $k = 2.5\%$ of n
  - $H_0 : \alpha_{\text{post}} = \alpha_{\text{pre}}$ against $H_1 : \alpha_{\text{post}} > \alpha_{\text{pre}}$
  - $z = (\hat{\alpha}_{\text{post}} - \hat{\alpha}_{\text{pre}}) / \sqrt{s_{\text{pre}}^2 + s_{\text{post}}^2}$, with s the moving-block bootstrap standard errors (blocks of 20 days)
  - under H_0, $z \approx N(0,1)$; reject H_0 at 5% if $z > 1.645$
- Results
  - before (Sep 2014 – 10 Jan 2024, $k = 85$): $\hat{\alpha} = 4.16$ ($s = 0.52$)
  - after (11 Jan 2024 – Sep 2026, $k = 24$): $\hat{\alpha} = 4.31$ ($s = 0.91$)
  - $z = 0.15$
- Placebo: S&P 500, 17 Sep 2014–18 Sep 2026
  - break 11 Jan 2024, $k = 58/16$ before/after
  - $\hat{\alpha}$: $2.82 \to 2.86$, $z = 0.03$
- Type of evidence: description (it shows what happened in the data, without testing a cause)
  - a before/after comparison with low power (next slide), not the effect of the ETFs

MFM_ch1_ai_discovery

A Non-Significant Result with Low Power: What We Cannot Conclude

- **Minimum detectable change**, one-sided 5% test with 80% power
  - $(1.645 + 0.842) \times 1.05 = 2.6$: the rise in α the test detects with probability 80%
  - 1.645 and 0.842: the $N(0,1)$ quantiles for the one-sided 5% level and 80% power
  - $1.05 = \sqrt{0.52^2 + 0.91^2}$: the SE of the difference
  - the Normal-distribution approximation is rough with 24 tail observations
  - the data cannot distinguish an unchanged tail from a large thinning of the tail
- A non-significant result does not show the absence of an effect
  - wait for more data or pool assets

MFM_ch1_ai_discovery

Project Seed

- **Question**: did the tail index of Bitcoin and Ethereum change with institutionalisation?
  - institutionalisation: futures in 2017, spot ETFs in 2024
  - raw and GARCH-standardised returns (shape versus scale)
  - loss and gain tails separately
  - Bitcoin uses the course series; Ethereum is not part of the five-market panel (daily series: `data/market/ETH-USD.CC.csv`): document its source, the close time and time zone, and the currency (USD) before estimating
- **Replicate first**: the chapter's Hill table and the mini-case numbers
- **Pre-register**, committed before the results
  - H_0, H_1, the rule for k, break dates and bootstrap
- **Report**: the change-point test of Hoga (2017) as a Quantlet
  - estimates, both SEs, the minimum detectable change
- **Log**: prompts in `AI_USE.md`
  - every caught error (an i.i.d. SE, a date chosen from the data) in `AI_ERRORS.md`
  - reusable prompts (Appendix) and checks (Appendix)

What We Keep: the Size of Returns Is Forecastable, the Direction Hardly

Takeaways

- Autocorrelations of returns are generally small, with exceptions (S&P 500, EUR/RON), but their size is predictable
- Heavy tails are the rule: tail indices near 3
  - conditional on the tail approximation and the thresholds, the estimates cast doubt on a finite fourth moment
  - kurtosis-based inference is then unreliable
- Volatility is persistent (long memory or breaks)
  - in equity markets it reacts asymmetrically to losses
- Every test needs a reference distribution that the data may violate

Questions for the room

- Why does a mixture with σ_t drawn independently each day have heavy tails but no volatility clustering (Further Topics)?
- Why does the S&P 500 statistic fall from $Q(10) = 91.6$ to $\tilde{Q}(10) = 19.0$ with robust standard errors (Further Topics)?
- Why can the tail index be estimated with the Hill estimator when the kurtosis may not exist?

- Next chapter: are returns predictable at all? (Chapter 2)

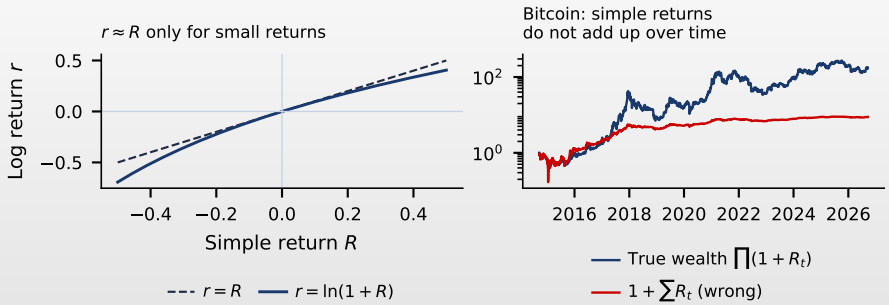
The Practical Decision

- **What I choose**
  - for tails: the Hill estimator ($k = 2.5\%$ of n) with block-bootstrap SEs, and quantiles (VaR 1%), not the kurtosis
  - for dependence: the robust bands, not the i.i.d. ones (S&P 500: 7 of 100 lags leave the robust band, 20 the i.i.d. one)
  - for modelling: time-varying volatility and heavy-tailed innovations (Chapter 5)
- **Under which conditions**
  - the tail is approximately Pareto and the Hill plot is flat around the chosen threshold
  - tests on r_t^2 and $|r_t|$ and the SEs of Sharpe ratios require a finite fourth moment, which the Hill estimates question
- **What result would make me drop it**
  - a single threshold: a Hill plot that does not settle (Bitcoin: 4.2 at $k = 2.5\%$; at a common threshold, 2.71 for losses and 3.36 for gains)
  - avoiding the kurtosis: bootstrap intervals for α lying wholly above 4 on every market
  - heavy-tailed innovations: a kurtosis of $\hat{z}_t = r_t/\hat{\sigma}_t$ close to 3

Why the Log Return Is Always Below the Simple Return

- $r = \ln(1+R) < R$ for every $R \neq 0$, because $\ln(1+x) < x$ for $x > -1$, $x \neq 0$ (the logarithm is concave)
- The gap is negligible for small moves and large for big falls
  - $R = +1\%$ gives $r = 0.995\%$
  - $R = -50\%$ gives $r = -69.3\%$
- Example: the price goes from 100 to 110, then to 99 (Seminar 1, A1)
  - the simple returns $+10\%$ and -10% sum to 0, yet the value falls by 1%: $1.1 \times 0.9 = 0.99$
  - the log returns 9.53% and -10.54% sum to $-1.01\% = \ln 0.99$
- Textbooks: Campbell, Lo & MacKinlay (1997), ch. 1; Franke, Härdle & Hafner (2019)

Simple and Log Returns in Practice



- Left: $r \approx R$ only for small moves
- Right: Bitcoin 2014–2026 multiplied wealth by 176.9, i.e. $\sum_t r_t = \ln 176.9 = 5.18$
  - the sum of daily log returns is $\ln(P_T/P_0)$, the log of the multiple
  - adding daily simple returns gives $1 + \sum_t R_t = 8.83$, 20 times below the true multiple

MFM_ch1_returns

How Precisely We Estimate the Mean, Skewness and Kurtosis

- **Standard error** (SE) of the mean: $\mathrm{SE}(\bar{r}) = \sigma/\sqrt{T}$, estimated with $\hat{\sigma}$
  - daily means are 13–40 times smaller than $\hat{\sigma}$: **relative to its size**, the mean is the least precisely estimated moment
  - S&P 500: $\bar{r} = 0.033\% \pm 0.023\%$ a day (95% confidence interval, $\pm 1.96\,\mathrm{SE}$), $T = 9{,}245$; $\mathrm{SE}/|\bar{r}| \approx 0.36$
- i.i.d. data from the Normal distribution: $\sqrt{T}\,\hat{S} \to N(0,6)$ and $\sqrt{T}(\hat{K}-3) \to N(0,24)$
  - so $\mathrm{SE}(\hat{S}) \approx \sqrt{6/T}$ and $\mathrm{SE}(\hat{K}) \approx \sqrt{24/T}$: 0.025 and 0.051 for the S&P 500
- Higher moments have a different problem: their SE may not exist at all (section on tails)

The Delta Method and the Moments That Must Exist

- The **delta method** gives the standard error of a function of an estimator

$$\sqrt{T}(\hat{\theta} - \theta) \to N(0, v) \implies \sqrt{T}\left(g(\hat{\theta}) - g(\theta)\right) \to N(0, g'(\theta)^2 v)$$

  - $\hat{\theta}$: the estimator of a parameter θ (e.g. a moment), with asymptotic variance v
  - g: a differentiable function, e.g. the ratio of moments that defines $\hat{S}$
  - the idea: the first-order Taylor expansion $g(\hat{\theta}) \approx g(\theta) + g'(\theta)(\hat{\theta} - \theta)$
- For i.i.d. data from any distribution the limit is Normal only if the moments in its variance exist
  - $\hat{S}$: $\mathbb{E}r^6 < \infty$; $\hat{K}$: $\mathbb{E}r^8 < \infty$ ($\mathbb{E}$: the expectation)
  - whether they exist is checked empirically in the section on tails

What Each Test Assumes and Where It Fails on Returns

Test	H_0	χ^2 limit needs	Problem here
JB, Jarque & Bera (1980)	i.i.d., Normal distribution	H_0 itself (all moments exist)	with heavy tails, near-certain rejection for large T
LB, Ljung & Box (1978)	$\rho_1 = \cdots = \rho_m = 0$	i.i.d., not only uncorrelated	volatility clustering: over-rejects
ARCH-LM, Engle (1982)	i.i.d., no ARCH effects	under the i.i.d. H_0: $\mathbb{E}r^4 < \infty$ ($\alpha > 4$)	$\alpha \approx 3 < 4$: size unreliable

- Two moment conditions for ARCH-LM, in two different situations
  - under the i.i.d. null: the χ^2_q limit needs a finite variance of r_t^2, i.e. $\mathbb{E}r^4 < \infty$
  - under GARCH-type dependence: the sample ACF of r_t^2 behind the statistic uses a standard Normal limit under the moment condition $\mathbb{E}r^8 < \infty$ ($\alpha > 8$) Davis & Mikosch (1998)
- The question is not how to compute the statistic, but whether its reference distribution holds for returns

Worked Example: JB for the S&P 500

- 5% critical value of χ^2_2: 5.99
- S&P 500, 1990–2026: $T = 9{,}245$, $\hat{S} = -0.364$, $\hat{K} - 3 = 10.88$
  - skewness term $\hat{S}^2/6 = 0.132/6 = 0.022$
  - kurtosis term $10.88^2/24 = 118.4/24 = 4.93$
  - $JB = 9{,}245 \times (0.022 + 4.93) \approx 45{,}800$, against the critical value 5.99
- Almost all of the statistic comes from the kurtosis term
  - the problem is the tails, not the asymmetry

What a JB Rejection Does and Does Not Show

- Under the null all moments exist
  - so the p-value is valid for the **joint** null "i.i.d. and Normal distribution"
  - heavy tails make JB grow with T
  - this is power, not a broken p-value
- A rejection does not say which part fails
  - it may be the marginal distribution or independence
  - tests of the marginal distribution with HAC (heteroskedasticity and autocorrelation consistent) variances:

Appendix

Robust Alternative: Quantile-Based Skewness and Kurtosis

- Two quantile-based measures Kim & White (2004), which need no moments
  - q_p: the quantile of order p of returns, the value below which a share p of days lies; $q_{0.5}$: the median
- **Hinkley skewness** compares the lengths of the two tails around the median

$$\frac{q_{0.95} + q_{0.05} - 2q_{0.5}}{q_{0.95} - q_{0.05}}$$

  - the numerator is $(q_{0.95} - q_{0.5}) - (q_{0.5} - q_{0.05})$: right tail minus left tail
  - it lies in $[-1, 1]$; a negative value indicates a longer left tail
- **Crow–Siddiqui excess kurtosis**, denoted K_{CS}, compares the width of the tails with the width of the centre

$$K_{CS} = \frac{q_{0.975} - q_{0.025}}{q_{0.75} - q_{0.25}} - 2.91$$

  - 2.91: the value of the ratio for the Normal distribution; a positive value indicates heavier tails
  - both measures are 0 for the Normal distribution

Confidence Intervals from a Moving-Block Bootstrap

- 95% confidence intervals (CI) from a **moving-block bootstrap**
  - it resamples blocks of 20 consecutive days (1,000 resamples)
  - the measure is recomputed on each resampled series
  - the blocks keep the dependence within them
- Valid for stationary, weakly dependent returns with a positive density at these quantiles
- A CI that excludes 0 indicates a significant departure from the Normal distribution

Heavy Tails Are Confirmed without Moments

	Moment skew.	Hinkley [95% CI]	Excess kurt.	Crow–Siddiqui excess [95% CI]
S&P 500	−0.36	−0.064 [−0.092, −0.041]	10.9	1.54 [1.29, 1.79]
BET-TR	−1.40	−0.036 [−0.085, 0.025]	17.8	1.06 [0.73, 1.43]
Bitcoin	−0.70	−0.018 [−0.069, 0.032]	12.0	2.25 [1.88, 2.64]
EUR/RON	0.73	0.041 [−0.022, 0.093]	15.1	4.87 [3.74, 6.13]
Gold	−0.23	0.004 [−0.025, 0.030]	9.3	1.47 [1.29, 1.73]

- Quantile (Crow–Siddiqui) excess kurtosis is positive in all five markets
  - no CI contains 0
- Quantile-based negative skewness is established only for the S&P 500
  - BET-TR's moment skewness −1.40 comes from a few days (19 December 2018)

MFM_ch1_inference

The Student-t Distribution: a Heavy-Tailed Model

- The Student-t distribution, Praetz (1972): a heavy-tailed model for returns, with density

$$f(x) \propto \left(1 + \frac{(x-\mu)^2}{\nu s^2}\right)^{-(\nu+1)/2}$$

  - $\propto$: without the normalising constant; μ: the centre; $s > 0$: the scale
  - $\nu > 0$: the degrees of freedom; a small ν means heavy tails, and $\nu \to \infty$ gives the Normal distribution
- Tail $P(X > x) \propto x^{-\nu}$: moments exist only of order $< \nu$
  - variance $s^2\nu/(\nu-2)$ for $\nu > 2$
  - excess kurtosis $6/(\nu-4)$ for $\nu > 4$

Extreme Days: Observed, under the Normal Distribution and under the Student-t

	S&P 500	BET-TR	Bitcoin	EUR/RON	Gold
Days beyond $3\hat{\sigma}$ (%)	1.55	1.44	1.82	2.06	1.42
Days beyond $5\hat{\sigma}$ (count)	30	12	10	29	26
Expected under the Normal distribution, $> 5\hat{\sigma}$	0.005	0.002	0.003	0.003	0.005
Student-t degrees of freedom $\hat{\nu}$ (maximum likelihood)	2.8	3.1	2.2	1.2	2.9

- Under the Normal distribution
  - 0.27% of days lie beyond 3σ (observed 5–8 times more)
  - one 5σ day occurs every 6,900 years of trading (observed 10–30)
- $\hat{\nu}$ is fitted by **maximum likelihood** to the whole distribution
  - it picks the μ, s and ν that make the observed data most probable
  - the many small returns dominate the fit, so a poor fit in the centre moves $\hat{\nu}$
  - EUR/RON: $\hat{\nu} = 1.2$ reflects that 57% of daily changes are below 0.1%, plus rare jumps

Choosing k: between Variance and Bias

- **Choice of** k, i.e. of the threshold $u = X_{(k+1)}$
  - small k: few tail observations, so a large variance (SE $\approx \alpha/\sqrt{k}$)
  - large k: observations from the centre enter, where the tail is no longer Pareto, so $\hat{\alpha}$ is biased
  - read the level where the **Hill plot** ($\hat{\alpha}_k$ against k) is flat
- The chapter's rule, fixed before estimation: $k = 2.5\%$ of n
  - the middle of the 1–5% range where the Hill plots are roughly flat
- Robustness: we also report the smallest and the largest $\hat{\alpha}_k$ for k between 1% and 5% of n
- The Student-t has tail index $\alpha = \nu$, but fitted to the whole distribution it misdescribes the tails
  - for EUR/RON $\hat{\alpha} = 2.8$ (Hill) while $\hat{\nu} = 1.2$

The ACF under Conditional Heteroskedasticity

- **ACF** (AutoCorrelation Function): the sample autocorrelations $\hat{\rho}_k$, plotted against the lag k

$$\hat{\rho}_k = \frac{\sum_t x_t x_{t-k}}{\sum_t x_t^2}, \qquad x_t = r_t - \bar{r}$$

- Uncorrelated returns whose conditional variance moves: $\hat{\rho}_k \approx N(0, \tau_k)$ Romano & Thombs (1996)
  - τ_k: the variance of $\hat{\rho}_k$, estimated by $\hat{\tau}_k = \sum_t x_t^2 x_{t-k}^2 / (\sum_t x_t^2)^2$
  - i.i.d.: $T\tau_k = 1$, which gives the classical band $\pm 1.96/\sqrt{T}$
  - volatility clustering: $T\tau_k > 1$; the classical band is too narrow and flags autocorrelations as significant too often

The Robust Portmanteau Test $\tilde{Q}(m)$

- Robust t-statistic: $\hat{\rho}_k / \sqrt{\hat{\tau}_k}$, compared with ± 1.96
- Robust portmanteau test Lobato, Nankervis & Savin (2001): null hypothesis $\rho_1 = \cdots = \rho_m = 0$

$$\tilde{Q}(m) = \sum_{k=1}^{m} \frac{\hat{\rho}_k^2}{\hat{\tau}_k}$$

  - the sum of the squared robust t-statistics for the first m lags; reference distribution χ^2_m
  - its assumptions: Appendix
- The difference from the Ljung–Box $Q(m)$
  - $Q(m)$ divides each $\hat{\rho}_k^2$, approximately, by the i.i.d. variance $1/T$
  - $\tilde{Q}(m)$ divides by the estimated variance $\hat{\tau}_k$

With Robust Standard Errors, Only the S&P 500 and EUR/RON Stay Significant

Lag 1	S&P 500	BET-TR	Bitcoin	EUR/RON	Gold
$\hat{\rho}_1$	−0.079	0.061	−0.022	0.170	−0.011
$T\hat{\tau}_1$	5.00	4.35	2.89	6.85	2.43
t-statistic, i.i.d. / robust	−7.6 / −3.4	3.3 / 1.6	−1.5 / −0.9	12.4 / 4.7	−1.1 / −0.7

- S&P 500: $Q(10) = 91.6$ but $\tilde{Q}(10) = 19.0$ (p $= 0.041$)
- With robust SEs only the S&P 500 ($t = -3.4$) and EUR/RON ($t = 4.7$) keep a significant lag-1 autocorrelation
  - it is economically small

MFM_ch1_inference

Heavy Tails from Random Volatility: Mixtures of Normal Distributions

- The model: the return is a Normal shock times a random volatility

$$r_t = \sigma_t \varepsilon_t, \qquad \varepsilon_t \sim N(0,1) \text{ independent of } \sigma_t$$

- The kurtosis of returns is at least 3

$$K = \frac{\mathbb{E}[r_t^4]}{\mathbb{E}[r_t^2]^2} = 3\,\frac{\mathbb{E}[\sigma_t^4]}{\mathbb{E}[\sigma_t^2]^2} \;\geq\; 3$$

  - the middle equality: $\mathbb{E}[\varepsilon_t^4] = 3$ and $\mathbb{E}[\varepsilon_t^2] = 1$, and σ_t and ε_t are independent
  - the inequality is Jensen's inequality ($\mathbb{E}[X^2] \geq \mathbb{E}[X]^2$) with $X = \sigma_t^2$
  - equality only if σ_t is constant; any variation in volatility creates excess kurtosis

A Numerical Example: Kurtosis 8.33 from Normal Shocks

- Example: $\sigma = 1\%$ on 90% of days and 3% on 10% of days

$$K = 3 \cdot \frac{0.9 \cdot 1 + 0.1 \cdot 81}{(0.9 + 0.1 \cdot 9)^2} = 3 \cdot \frac{9.0}{3.24} = 8.33$$

  - numerator: $\mathbb{E}[\sigma^4] = 0.9 \cdot 1^4 + 0.1 \cdot 3^4$, in units of $(1\%)^4$
  - denominator: $\mathbb{E}[\sigma^2]^2$, with $\mathbb{E}[\sigma^2] = 0.9 \cdot 1^2 + 0.1 \cdot 3^2 = 1.8$
  - the excess kurtosis is 5.33, although each day's return, given σ, is Normally distributed
- This links heavy tails (Fact 2) to volatility clustering (Fact 6)
  - a mixture drawn independently every day has heavy tails but no clustering (Seminar 1, A4 and A9)

Estimating the Memory Parameter d

- Long memory in the frequency domain: the spectral density grows without bound at low frequencies

$$f(\lambda) \approx G\lambda^{-2d} \quad \text{as } \lambda \to 0$$

  - $f(\lambda)$: the spectral density, the contribution of cycles of frequency λ to the variance of the series; small λ: long cycles
  - $G > 0$: a constant; semiparametric estimators use only the m lowest Fourier frequencies $\lambda_j = 2\pi j/T$, $j = 1, \ldots, m$

- Two estimators (formulas in the Appendix)
  - local Whittle Robinson (1995): $\sqrt{m}(\hat{d} - d) \to N(0, \frac{1}{4})$, so SE $= 1/(2\sqrt{m})$
  - log-periodogram regression Geweke & Porter-Hudak (1983): SE $= \pi/\sqrt{24m}$
- The choice $m = T^{0.65}$
  - a small m inflates the SE
  - a large m takes in frequencies where short-run (GARCH) dynamics bias $\hat{d}$

Long Memory or Structural Breaks? (1/2)

- Rare shifts in the level of volatility produce a slowly decaying ACF of $|r_t|$
  - and $\hat{d} > 0$ without long memory Diebold & Inoue (2001), Mikosch & Stărică (2004)
- Check: locate one variance break with the cumulative sum of squares Inclán & Tiao (1994)
  - then re-estimate d (local Whittle) on each side

$\lvert r_t \rvert$	Break date	Volatility before / after (%)	$\hat{d}$ before (SE)	$\hat{d}$ after (SE)
S&P 500	20 Jul 1998	12.6 / 19.3	0.28 (0.04)	0.51 (0.03)
BET-TR	17 Dec 2018	12.5 / 16.9	0.19 (0.05)	0.37 (0.04)
Bitcoin	19 Jun 2022	74.9 / 47.7	0.39 (0.04)	0.15 (0.05)
EUR/RON	27 Jun 2013	6.6 / 2.2	0.35 (0.04)	0.35 (0.04)
Gold	29 Nov 2005	13.2 / 18.3	0.30 (0.03)	0.37 (0.03)

MFM_ch1_inference

Long Memory or Structural Breaks? (2/2)

- $\hat{d}$ stays at least 3 SEs above 0 in every sub-sample
  - weakest: Bitcoin after 2022, $0.15/0.05 = 3$
  - one break does not explain the memory
- S&P 500 after 1998: $\hat{d} = 0.51$, above the stationarity boundary $\frac{1}{2}$
  - but within one SE of it
  - interpret with care
- $\hat{d}$ moves (Bitcoin $0.39 \to 0.15$)
  - several breaks or a time-varying d remain open
  - the break dates are descriptive

MFM_ch1_inference

Which Inference Remains Valid When Moments Are Missing

Statistic	Needs	With Hill $\hat{\alpha} \approx 2.3$–3.5 (Bitcoin: 4.2)
Mean, variance of r_t	$\alpha > 2$	finite, slow convergence
ACF of $\lvert r_t \rvert$, robust ACF bands, $\tilde{Q}$	$\alpha > 4$ for Normal limits	doubtful
Kurtosis, ACF of r_t^2, ARCH-LM, Q on r_t^2	$\alpha > 4$ to exist (and for ARCH-LM's χ^2 under i.i.d.), $\alpha > 8$ for Normal limits under GARCH-type dependence	no standard inference
Skewness	$\alpha > 6$ for a Normal limit	no standard inference
Quantiles, number of tail days, Hill $\hat{\alpha}$	no moments	valid, with dependence-robust SEs

- With an infinite fourth moment the sample ACF of r_t^2 converges slowly
  - its limit is not the Normal distribution Davis & Mikosch (1998)
  - the $\pm 1.96/\sqrt{T}$ band and the p-values of Q on r_t^2 and ARCH-LM are not reliable
- The conclusion "volatility clusters" is robust
  - the exact magnitudes are not robust
  - one reason behind the Taylor effect: $\lvert r_t \rvert$ needs fewer finite moments than r_t^2

The Garman–Klass Estimator: the Range plus the Open-to-Close Move

- **Garman–Klass**, Garman & Klass (1980): combines the range with the open-to-close move

$$\hat{\sigma}^2_{GK} = \frac{1}{n}\sum_{t=1}^{n}\left[\tfrac{1}{2}(h_t - l_t)^2 - (2\ln 2 - 1)(c_t - o_t)^2\right]$$

  - $c_t - o_t$: the open-to-close log return
  - the weights $\frac{1}{2}$ and $2\ln 2 - 1 \approx 0.386$ come from the minimum-variance estimator derived by Garman and Klass
  - the same assumptions as Parkinson: no drift, no overnight jumps
- Efficiency relative to close-to-close: 7–8

The Rogers–Satchell Estimator: Unbiased under a Drift

- **Rogers–Satchell**, Rogers & Satchell (1991): unbiased under a non-zero **drift** (expected trend of the log price)

$$\hat{\sigma}^2_{RS} = \frac{1}{n}\sum_{t=1}^{n}[(h_t - o_t)(h_t - c_t) + (l_t - o_t)(l_t - c_t)]$$

  - the terms measure how far the high and the low moved away from the open and from the close
  - both products are ≥ 0, because $l_t \leq o_t, c_t \leq h_t$
  - for a Brownian motion with drift their mean is exactly the daily variance
  - it does not include the overnight jump

The Yang–Zhang Estimator: Adding the Overnight Jump

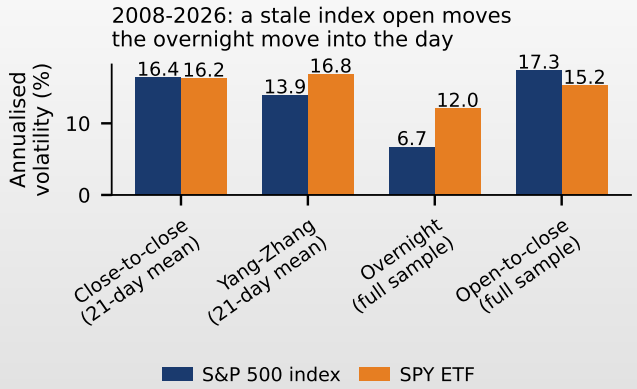
- **Yang–Zhang**, Yang & Zhang (2000): adds the overnight (close-to-open) jump

$$\hat{\sigma}^2_{YZ} = \hat{\sigma}^2_{\text{overnight}} + k\,\hat{\sigma}^2_{\text{open-close}} + (1-k)\,\hat{\sigma}^2_{RS}, \quad k = \frac{0.34}{1.34 + \frac{n+1}{n-1}}$$

  - $\hat{\sigma}^2_{\text{overnight}}$: the sample variance of $o_t - c_{t-1}$, the overnight return
  - $\hat{\sigma}^2_{\text{open-close}}$: the sample variance of $c_t - o_t$
  - k: the weight that gives the minimum variance among drift-independent estimators robust to opening jumps (not the Hill k); $k \approx 0.145$ for large n

- Range estimators are the bridge to realised volatility from intraday data (Chapter 9)

Stale Opens: the S&P 500 Index versus SPY, an Exchange-Traded Fund (ETF)



MFM_ch1_seminar_tests

Stale Opens: the Cause and the Effect on Volatility Estimators

- **Stale open**: the index open is computed from the constituents' first prices
  - at 9:30 many stocks have not traded yet, so their previous close enters the open
- SPY, the ETF that tracks the S&P 500, opens at a traded price
- Open = previous close
  - about 96% of days in 1996–2005 and still 16% in 2007
  - 3.5% over 2008–2026 (up to 26% in 2012–2013): range-based estimators therefore use data from 2008 on
- A stale open moves part of the overnight move into the day
  - overnight volatility: 6.7% for the index, 12.0% for SPY
- Yang–Zhang falls below close-to-close on the index only
  - use a traded instrument for estimators that use the open (Seminar 1, B15)

MFM_ch1_seminar_tests

The Maximum Drawdown against a Gaussian Random Walk

	S&P 500	BET-TR	Bitcoin	EUR/RON	Gold
Observed D (MDD)	0.84 (−57%)	0.37 (−31%)	1.80 (−83%)	0.17 (−16%)	0.59 (−45%)
$\mathbb{E}[D]$, $\mu = 0$	1.37	0.67	2.89	0.26	1.24
$\mathbb{E}[D]$, $\mu = \hat{\mu}$: formula / simulation	0.65 / 0.66	0.26 / 0.26	1.48 / 1.55	0.14 / 0.15	0.62 / 0.64
p-value, $P(D \geq D_{\text{obs}})$	0.17	0.07	0.26	0.29	0.52

- The driftless benchmark overstates $\mathbb{E}[D]$ about twofold
  - a positive drift limits how deep prices fall
- With the estimated drift no MDD is unusual: p-values from 0.07 (BET-TR) to 0.52 (gold)
- The benchmark ignores heavy tails and volatility clustering, so it is only a first point of comparison
- The MDD grows with the sample length: compare only equal horizons (Appendix)

generate_all_charts

Recovery Duration after the Maximum Drawdown

- **Recovery duration**: the time from the peak until the price reaches it again

	MDD	Peak	Trough	Peak regained
S&P 500	−57%	Oct 2007	Mar 2009	Mar 2013 (5.5 years)
BET-TR	−31%	Jan 2020	Mar 2020	Dec 2020 (0.9 years)
Bitcoin	−83%	Dec 2017	Dec 2018	Nov 2020 (3.0 years)
EUR/RON	−16%	Dec 2005	Jul 2007	Jan 2008 (2.0 years)
Gold	−45%	Sep 2011	Dec 2015	Jul 2020 (8.9 years)

- Depth and duration are different risks
  - gold's drawdown was shallower than Bitcoin's but lasted three times longer
- Fast V-shaped recoveries (BET-TR 2020)
  - versus long underwater periods (gold 2011–2020)
- For EUR/RON a "drawdown" is a fall of the euro against the leu

MFM_ch1_risk_measures

The Sortino and Calmar Ratios: Losses and Drawdown in the Denominator

- **Sortino ratio** Sortino & van der Meer (1991): the Sharpe numerator, with only the volatility of losses in the denominator

$$SoR = \frac{\mu_{\text{arith}} - r_f}{\sigma^-_{\text{ann}}}, \qquad \sigma^- = \sqrt{\frac{1}{T}\sum_{t=1}^{T} \min(R_t, 0)^2}, \qquad \sigma^-_{\text{ann}} = \sqrt{A}\,\sigma^-$$

  - σ^-: the daily downside deviation (in the seminar: the downside semi-deviation $\hat{\sigma}_D$); the average runs over all T days, positive days count as 0
  - S&P 500: $\sigma^-_{\text{ann}} = 12.68\%$, $SoR = 0.78$

- **Calmar ratio**: CAGR/$|MDD|$
  - S&P 500: $8.69\%/56.78\% = 0.15$, a value driven by one episode
  - no usable sampling theory: the maximum drawdown in the denominator grows with the sample length

BET-TR versus S&P 500: the Sharpe Difference Is Not Significant at 5%

- Clustering: HAC or studentised block-bootstrap tests Ledoit & Wolf (2008)
  - they still need finite fourth moments, which the Hill estimates put in doubt
- BET-TR versus S&P 500, common days 2014–2026
  - difference 0.61, $z = 1.75$ ($p = 0.08$)
  - block-bootstrap 95% CI $[-0.01, 1.21]$ (Seminar 1, B14)

MFM_ch1_risk_measures

Case Study (1/7): One Model for Volatility Clustering, Long Memory and Leverage

- Requirement: a single model must reproduce simultaneously
  - volatility clustering (Fact 6)
  - the slow decay of the autocorrelation of $|r_t|$ (Fact 8)
  - the leverage effect (Fact 9)
- Why it matters: if volatility is mostly a function of the price path, it is mostly endogenous
  - a **path-dependent volatility (PDV)** model needs little extra randomness
  - unlike stochastic volatility models (Chapters 11–12)
- Data: daily closes of five indices
  - S&P 500 (US), Euro Stoxx 50 (euro area), FTSE 100 (UK), DAX (Germany), Nikkei 225 (Japan)
  - training 1 Jan 2000 – 31 Dec 2018
  - test 1 Jan 2019 – 15 May 2022

Case Study (2/7): Volatility as a Function of the Trend and of Recent Volatility

- Equation (3.7) in the paper: today's volatility is a linear function of two summaries of past returns

$$\sigma_t = \beta_0 + \beta_1 R_{1,t} + \beta_2 \Sigma_t$$

- Notation
  - σ_t: the dependent variable, for example the VIX at the close of day t
  - S_{t_i}: the index level at the close of day t_i; $t_i \leq t$ runs over the days up to and including t
  - $r_{t_i} = S_{t_i}/S_{t_{i-1}} - 1$: the **simple** return of day t_i (the paper's notation; elsewhere in this chapter r_t is the log return)
  - $R_{1,t}$: the trend component, built from past returns
  - Σ_t: the volatility component, built from past squared returns

Case Study (3/7): How Much Does an Old Day Count?

- A **kernel** $K(\tau)$ gives the weight of a day of age τ, with $\tau = i\Delta t$ for a day i days old and $\Delta t = 1/252$
- The daily weights $K(i\Delta t)\Delta t$ are positive and sum to 1 (the vertical axis of the kernel chart)
- The paper uses a **time-shifted power law (TSPL)**, eq. (3.9)

$$K(\tau) = \frac{(\tau + \delta)^{-\alpha}}{Z_{\alpha,\delta}}, \qquad \alpha > 1,\ \delta > 0,\ \tau = i\Delta t,\ i = 0, 1, \ldots, 999$$

  - $Z_{\alpha,\delta}$: the normalising constant over the last 1,000 days
  - α: how fast the weights decay; a small α means slowly decaying weights
  - δ: the time shift, in years; a small δ concentrates the weight on the last few days ($\delta = 0.05$ years ≈ 12.6 days)
- Resulting shape: large weight on the last few days and a slowly decaying power-law tail
  - large weight on the last days: short memory
  - power-law tail: slowly decaying weights, consistent with the slow decay of the ACF of $|r_t|$ (Fact 8)
  - truncated at 1,000 days, the kernel does not by itself show asymptotic long memory
- The model uses two kernels: K_1 for the trend and K_2 for volatility, with parameters (α_1, δ_1) and (α_2, δ_2)
- Here α, δ and K are not the tail index, the Taylor-effect power or the kurtosis

Case Study (4/7): The Recent Trend and a Weighted Historical Volatility

- Trend component: a weighted sum of past returns

$$R_{1,t} = \sum_{t_i \le t} K_1(t - t_i)\, r_{t_i}$$

  - $t - t_i$: the age of day t_i, measured in years as in the paper (one day $= 1/252$ years); $K_1(t - t_i)$: the weight of that day
  - $R_{1,t} < 0$ after a run of falls, $R_{1,t} > 0$ after a run of rises
- Volatility component: the square root of a weighted sum of squared returns

$$\Sigma_t = \Big(\sum_{t_i \le t} K_2(t - t_i)\, r_{t_i}^2 \Big)^{1/2}$$

  - The weights $K_j(i\Delta t)\Delta t$ sum to 1: $R_{1,t}$ and Σ_t are weighted means, annualised with 252 and $\sqrt{252}$
- Interpreting the coefficients
  - $\beta_1 < 0$: volatility rises after falls ($R_{1,t} < 0$), i.e. the leverage effect
  - $\beta_2 > 0$: high recent volatility raises today's volatility, i.e. volatility clustering, as in GARCH
  - $\hat{\beta}_2 = 0.87 < 1$, an empirical finding, not a constraint: the VIX moves less than one-for-one with Σ_t
  - $\beta_0 > 0$ supplies part of the level

Case Study (5/7): Seven Parameters, Scored Also out of Sample

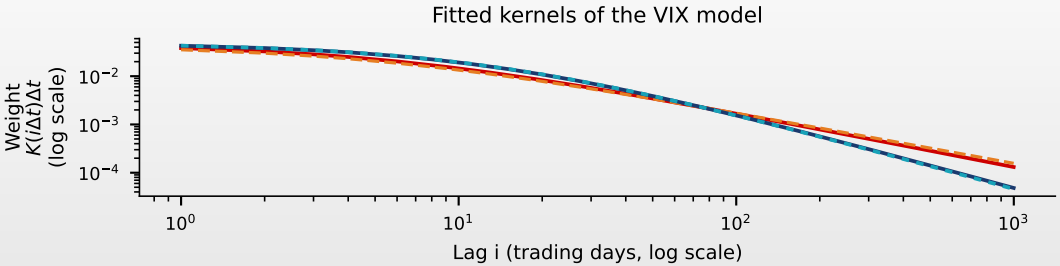
- Seven parameters: $\beta_0, \beta_1, \beta_2, \alpha_1, \delta_1, \alpha_2, \delta_2$
  - estimated by nonlinear least squares on the training set: minimise $\sum_t (\sigma_t - \hat{\sigma}_t)^2$
  - $\hat{\sigma}_t$: the model value for day t
  - estimation details: Appendix
- Scores, computed separately on the training and the test set
  - RMSE (root mean squared error) $= \sqrt{\frac{1}{T}\sum_t (\sigma_t - \hat{\sigma}_t)^2}$: the typical error, in volatility units (decimals: 0.01 = 1 VIX point)
  - $R^2 = 1 - \sum_t (\sigma_t - \hat{\sigma}_t)^2 / \sum_t (\sigma_t - \bar{\sigma})^2$, with $\bar{\sigma}$ the mean of σ_t

Case Study (6/7): After 2022, a Smaller Error but Also a Lower R^2

- Parameters fitted on the training period, 1 Jan 2000 – 31 Dec 2018 (4,779 trading days), then frozen
- The test set is shaded
- After the test set, 16 May 2022 – 18 Sep 2026: $R^2 = 0.740$, RMSE 0.026
  - a lower RMSE, but the VIX varied less (standard deviation 5.0 versus 9.3 points), so R^2 falls
- Ratio observed/fitted
  - mean 0.85 in April–May 2020
  - mean 1.16 from October 2020 to May 2022
  - 8 April 2025: VIX 52.3 versus 39.2 fitted

MFM_ch1_path_dependent_vol

Case Study (7/7): Fitted Kernels



MFM_ch1_path_dependent_vol

Formula Sheet with Assumptions

Returns and annualisation

- $R_t = P_t/P_{t-\mathbf{1}} - 1$, $r_t = \ln(1+R_t)$, $r_t(k) = \sum_{j=\mathbf{0}}^{k-\mathbf{1}} r_{t-j}$, $R_{p,t} = \sum_i w_i R_{i,t}$
- $\text{CAGR} = (P_T/P_{\mathbf{0}})^{\mathbf{1}/Y} - 1$
- $\sigma_{\mathbf{ann}} = \sqrt{A}\,\hat{\sigma}$: uncorrelated returns, constant variance

Tests

- $JB = T\left[\hat{S}^{\mathbf{2}}/6 + (\hat{K}-3)^{\mathbf{2}}/24\right]$: $\chi^{\mathbf{2}}_{\mathbf{2}}$ under i.i.d. Normal
- $Q(m) = T(T+2)\sum_{k=\mathbf{1}}^{m} \hat{\rho}_k^{\mathbf{2}}/(T-k)$: $\chi^{\mathbf{2}}_m$ under i.i.d.
- $\tilde{Q}(m) = \sum_k \hat{\rho}_k^{\mathbf{2}}/\hat{\tau}_k$: $\mathbb{E}r^{\mathbf{4}} < \infty$, zero cross terms
- Hill $\text{SE}(\hat{\alpha}) \approx \hat{\alpha}/\sqrt{k}$: independent Pareto tail

Volatility

- Parkinson $\frac{\mathbf{1}}{\mathbf{4}\ln \mathbf{2}}\overline{(h-l)^{\mathbf{2}}}$, the bar a mean over days: driftless, no overnight jumps
- Mixture $K = 3\,\mathbb{E}[\sigma^{\mathbf{4}}]/\mathbb{E}[\sigma^{\mathbf{2}}]^{\mathbf{2}}$: σ_t independent of ε_t

Performance

- $SR = (\mu - r_f)/\sigma$ on simple returns: $\mu = A\bar{R}$, not the log mean $A\bar{r}$
  - $\ln(1+\text{CAGR}) \approx \mu - \sigma^{\mathbf{2}}/2$ (log-scale volatility drag)
  - $\text{SE} \approx 1/\sqrt{Y}$ for i.i.d. Normal returns
- $SoR = (\mu - r_f)/\sigma^{-}$
- $DD_t = P_t/\max_{s\le t} P_s - 1$; Calmar $= \text{CAGR}/|MDD|$

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Prices Before Computers, the Stock Ticker



Stock ticker designed by Thomas Edison

(Smithsonian collection)

Photo: Jaclyn Nash, Smithsonian National Museum of American History; public domain; Wikimedia Commons

- Before computers, trades reached brokers as prices printed on a paper **ticker tape**
- A daily close is one snapshot of this stream of prices
- The choice of snapshot (close, fixing, a midnight snapshot) matters (Chapter 0)
- Prices at the level of individual trades: market microstructure, Chapter 10

Back

The People Behind the Stylised Facts (1/2)



Carl Friedrich Gauss (1777–1855)

Painting: C.A. Jensen (1840); public domain; Wikimedia Commons

- The **Normal distribution**: the benchmark that returns violate



Louis Bachelier (1870–1946)

Photo: unknown author; public domain; Wikimedia Commons

- Prices as a random walk, in his thesis Bachelier (1900)



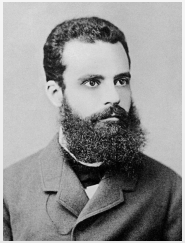
William S. Gosset, "Student" (1876–1937)

Photo: unknown author; public domain; Wikimedia Commons

- The Student-t distribution Student (1908), today a model for heavy tails

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The People Behind the Stylised Facts (2/2)



Vilfredo Pareto (1848–1923)

Photo: unknown author (1870s); public domain; Wikimedia Commons



Benoît Mandelbrot (1924–2010)

Photo: Rama (2007); CC BY-SA 2.0 France; Wikimedia Commons

- Pareto: **power laws** for large incomes, the ancestor of heavy-tail models
- Mandelbrot (1963): cotton prices
  - heavy tails, which the Normal distribution does not describe
- The exponent α of these power laws is the tail index estimated with the Hill method

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Cont's Facts in Later Studies

- Ratliff-Crain, Van Oort, Koehler & Tivnan (2025)
  - re-test all 11 facts on intraday returns of the Dow 30 stocks, October 2018 – March 2019
  - conclusive evidence for 8 facts, no support for the remaining 3
- Chakraborti, Toke, Patriarca & Abergel (2011)
  - survey the empirical facts of returns and of the order book, from intraday to lower frequencies
- Heavy tails beyond the Student-t: estimation of alpha-stable distributions, whose tail index is below 2 Pele (2014)
- Entropy-based measures of tail risk as an alternative to kurtosis Pele, Lazar & Dufour (2017)
- Crypto-assets: the same facts, with a tail factor that separates them from other asset classes Pele et al. (2023)

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Annual Variance under Autocorrelation (1/2)

- The variance of a sum of h daily returns depends on the autocorrelations

$$\mathrm{Var}\Big(\sum_{j=1}^{h} r_{t+j}\Big) = h\sigma^2\Big[1 + 2\sum_{j=1}^{h-1}(1 - j/h)\rho_j\Big]$$

  - h: the horizon in days; σ^2: the daily variance; ρ_j: the autocorrelation at lag j
  - assumption: weakly stationary returns (constant mean and variance, autocovariances depending only on the lag)
  - without autocorrelation ($\rho_j = 0$) it is $h\sigma^2$, i.e. the $\sqrt{h}$ rule for volatility
- **Variance ratio** $VR(h) = \mathrm{Var}(h\text{-day sum})/(h\sigma^2)$
  - $VR(h) > 1$ (< 1): net positive (negative) autocorrelation
  - volatility over horizon h is the $\sqrt{h}$-rule value multiplied by $\sqrt{VR(h)}$

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Annual Variance under Autocorrelation (2/2)

- Autoregression of order 1, AR(1), with $\rho_j = \rho^j$: $\sqrt{VR(h)} \to \sqrt{(1+\rho)/(1-\rho)}$ as h grows
  - ρ: the lag-1 autocorrelation, $|\rho| < 1$
  - EUR/RON, $\rho_1 = 0.17$ treated as an AR(1) parameter: factor 1.19 (4.4% → about 5.3%)
  - S&P 500, $\rho_1 = -0.079$: factor 0.92
- Using the sample lag-1 autocorrelation as an AR(1) parameter is an assumption
  - check the whole ACF (Seminar 1, A7)
  - EUR/RON: $\hat{\rho}_2, \hat{\rho}_3, \hat{\rho}_4 < 0$ offset lag 1, and monthly returns still give 4.4% (Chapter 0)
  - variance-ratio tests: Chapter 2

The Null Hypothesis of JB, and Tests for Time Series

- JB is a Lagrange Multiplier test Jarque & Bera (1980)
  - under i.i.d. returns from the Normal distribution, $\sqrt{T}\,(\hat{S}, \hat{K} - 3) \to N\big(0, \mathrm{diag}(6, 24)\big)$, hence χ^2_2
  - $\hat{S}$ and $\hat{K}$ are asymptotically independent, with variances $6/T$ and $24/T$: JB is the sum of the two squared standardised statistics
- GARCH-type dependence changes $\mathrm{Var}(\hat{S})$ and $\mathrm{Var}(\hat{K})$
  - a rejection mixes the marginal distribution and independence
- Bai & Ng (2005): tests of skewness, kurtosis and normality of the marginal distribution of a time series
  - studentised with HAC variances
  - their separate skewness and kurtosis tests (null $S = 0$ or $K = 3$ without normality) need $\mathbb{E}r^6 < \infty$ and $\mathbb{E}r^8 < \infty$

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The Robust Portmanteau Test and Its Assumptions

- $\tilde{Q}(m) = \sum_{k=1}^{m} \hat{\rho}_k^2 / \hat{\tau}_k \sim \chi_m^2$ Lobato, Nankervis & Savin (2001) needs $\mathbb{E}r^4 < \infty$
  - and zero cross terms: $\mathbb{E}[x_t^2 x_{t-k} x_{t-l}] = 0$ for two different lags $k \neq l$
  - then the $\hat{\rho}_k$ at different lags are asymptotically uncorrelated and the sum is χ_m^2
- Dalla, Giraitis & Phillips (2022) keep only the significant cross terms in the covariance of the $\hat{\rho}_k$
  - none for the S&P 500, BET-TR and Bitcoin
- $\hat{\tau}_k$ is the variance under $\rho_k = 0$
  - to compare a non-zero ρ_1 across periods use the HAC SE of the slope of r_t on r_{t-1} (Seminar 1, B11)

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Hill Estimator, Asymptotic Distribution and Alternatives

- Hill = maximum likelihood estimator of a Pareto tail above $u = X_{(k+1)}$, conditional on u (Seminar 1, A11)
- If $k \to \infty$ and $k/n \to 0$, and the tail is Pareto up to a second-order term

$$\sqrt{k}\,(\hat{\alpha}_k - \alpha) \;\to\; N(0, \alpha^2)$$

  - second-order term: a correction to the power law that vanishes as $x \to \infty$
  - consequence: $\mathrm{SE}(\hat{\alpha}_k) \approx \alpha/\sqrt{k}$, the formula in the main part
  - if k grows too fast, a bias term appears: the tail is only approximately Pareto
- Dependent data (GARCH-type): $\hat{\alpha}$ stays consistent
  - but extremes cluster and the variance exceeds α^2/k Hsing (1991)
- Choosing k: plot $\hat{\alpha}$ against $\log k$ ("alt-Hill" plot) Drees, Resnick & de Haan (2000)
  - the log scale stretches the region of small k, where the actual tail lies
- Regression alternative Gabaix & Ibragimov (2011)
  - log rank on log size, with rank shifted by $-\frac{1}{2}$: $\ln(i - \frac{1}{2}) = a - b \ln X_{(i)}$, $i = 1, \ldots, k$
  - the slope $\hat{b}$ estimates α; the shift by $\frac{1}{2}$ reduces the small-sample bias
  - $\mathrm{SE} = \hat{\alpha}\sqrt{2/k}$

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Two Tails and the Tail of $|r_t|$

- $P(|X| > x) = P(X > x) + P(X < -x)$
  - with indices α_- (losses) and α_+ (gains), $|X|$ has index $\min(\alpha_-, \alpha_+)$
- At the same k the estimates are not comparable
  - the k largest $|r_t|$ come from both tails
  - common threshold u: the lower of the two one-sided thresholds at $k = 2.5\%$ of n

	u (%)	k for $\lvert r_t \rvert$	$\hat{\alpha}$, $\lvert r_t \rvert$	$\hat{\alpha}_-$, losses	$\hat{\alpha}_+$, gains
S&P 500	2.17	508	2.90	2.85	2.96
BET-TR	1.79	153	2.54	2.20	3.06
Bitcoin	7.11	233	3.02	2.71	3.36
EUR/RON	0.54	295	2.28	2.63	2.04
Gold	1.98	513	3.11	2.97	3.28

- At a common u, the $|r_t|$ estimate lies between the two tails
  - Bitcoin's 4.2 at $k = 2.5\%$ reflects second-order bias
  - report both one-sided indices
  - Seminar 1, B6 uses $k = 2\%$ of n per tail: Bitcoin 2.89 / 3.62, BET-TR 2.19 / 3.08

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Which Tail Index Is the Chapter's Estimate?

- The chapter uses the fixed reference choice: $|r_t|$ at $k = 2.5\%$ of n (table in the main part)
- The common-threshold k for $|r_t|$ lies outside the 1–5% range of the main part
  - S&P 500 $508/9{,}245 = 5.5\%$, gold $513/9{,}559 = 5.4\%$, EUR/RON $295/5{,}352 = 5.5\%$, Bitcoin 5.3%, BET-TR 5.1%
  - a larger k lets less extreme days in, which can bias $\hat{\alpha}$ (choice of k)
- EUR/RON falls from 2.82 to 2.28
  - more than twice as many tail days (295 versus 133)
  - and the gain tail, $\hat{\alpha}_+ = 2.04$, is the heavier one: $|r_t|$ inherits the heavier tail
  - read the gap as threshold sensitivity, not as a second estimate of the same quantity
- The common-threshold table answers a different question
  - which tail, losses or gains, is heavier

Estimating d, the Formulas (1/2)

- Fourier frequencies $\lambda_j = 2\pi j/T$, $j = 1, \ldots, m$; the m lowest are used, here $m = T^{0.65}$
- The **periodogram**: the sample estimator of the spectral density

$$I(\lambda_j) = \frac{1}{2\pi T}\left|\sum_{t=1}^{T} x_t\, e^{-\mathrm{i}\lambda_j t}\right|^2$$

  - x_t: the demeaned series (here $|r_t|$ minus its mean)
  - $I(\lambda_j)$: how much of the variation in x_t comes from the cycle of frequency λ_j
- Log-periodogram regression Geweke & Porter-Hudak (1983): regress $\ln I(\lambda_j)$ on $-2\ln(2\sin(\lambda_j/2))$
  - the reason: $\ln f(\lambda) \approx \ln G - 2d \ln \lambda$, and $2\sin(\lambda/2) \approx \lambda$ at low frequencies
  - the slope is $\hat{d}$, SE $= \pi/\sqrt{24m}$

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Estimating d, the Formulas (2/2)

- Local Whittle Robinson (1995): $\hat{d}$ minimises an objective built from the approximation $f(\lambda) \approx G\lambda^{-2d}$

$$\hat{d} = \arg\min_{d} \left[\ln\left(\frac{1}{m} \sum_{j=1}^{m} \lambda_j^{2d} I(\lambda_j) \right) - \frac{2d}{m} \sum_{j=1}^{m} \ln \lambda_j \right]$$

  - the constant G has been eliminated analytically
  - $\sqrt{m}(\hat{d} - d) \to N(0, \frac{1}{4})$, so $\mathrm{SE} = 1/(2\sqrt{m})$
- Local Whittle is robust to non-Gaussianity
  - a larger m lowers the SE
  - but lets short-run (GARCH) dynamics bias $\hat{d}$ (Seminar 1, B12)

Markets in Detail

- **Crypto-assets**: an asset class with its own statistical profile Härdle, Harvey & Reule (2020)
  - their returns have little exposure to the risk factors of traditional markets Liu & Tsyvinski (2021)
  - tail, memory and moment factors classify cryptos as a separate asset class Pele et al. (2023)
  - crypto modelling and the CRIX (CRypto IndeX): Härdle, Chen & Overbeck (2017); Chapter 16
- **EUR/RON**: BNR reference rate from 1 July 2005
  - long calm periods interrupted by sudden adjustments
- **BET-TR**: the bank-asset tax was announced on 18 December 2018
  - adopted on 28 December 2018 as Government Emergency Ordinance (GEO) 114/2018
  - policy shocks produce single-day crashes without comparable rallies
- **Non-synchronous trading**: index constituents do not all trade at every moment
  - so the index reflects news with a delay
  - creating positive autocorrelation Campbell, Lo & MacKinlay (1997), ch. 3

Back

The Maximum Drawdown of a Brownian Motion

- Driftless Brownian motion of the log price with annual volatility σ over T years
  - here T is the horizon in years, not the number of days
- The maximum **log** drawdown $D = -\ln(1 + MDD) \geq 0$, the largest fall in the log price, has

$$\mathbb{E}[D] = \sqrt{\pi/2}\,\sigma\sqrt{T} \qquad \text{Magdon-Ismail, Atiya, Pratap \& Abu-Mostafa (2004)}$$

- $MDD = e^{-D} - 1$; $\sqrt{\pi/2} \approx 1.25$
  - with a positive drift $\mathbb{E}[D]$ grows only like $\ln T$ (formula in the main part)
- Gold 1990–2026: $\sigma = 16.3\%$, $T = 36.7$
  - without drift $\mathbb{E}[D] = 1.24$, twice the observed 0.592 (-44.7%) (Seminar 1, A2(b–e))
  - with the observed drift, $\mu = 6.5\%$ a year: $\mathbb{E}[D] \approx 0.62$ (formula) and 0.64 (simulation), close to the observed value
- Drawdowns and Calmar ratios from samples of different length are not comparable

Back

Sharpe Ratios beyond the Normal Distribution

- i.i.d., any distribution with $\mathbb{E}r^4 < \infty$ Opdyke (2007)
  - $\text{Var}(\widehat{SR}_d) \approx (1 - S\,SR_d + \frac{K-1}{4} SR_d^2)/T$
  - S, K: the skewness and kurtosis of daily returns; for the Normal distribution ($S = 0$, $K = 3$) this gives Lo's formula
  - negative skewness and high kurtosis raise the variance of $\widehat{SR}_d$ when $SR_d > 0$
  - here the SEs change by at most 0.02
- The Deflated Sharpe Ratio corrects for non-normality and multiple testing (Chapter 13)

	CAGR	Sharpe	Sortino	Calmar
S&P 500	8.7%	0.55	0.78	0.15
BET-TR	21.2%	1.34	1.87	0.68
Bitcoin	53.9%	0.98	1.45	0.65
EUR/RON	1.8%	0.43	0.65	0.12
Gold	6.7%	0.48	0.69	0.15

MFM_ch1_risk_measures

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Case Study, Estimation Details

- Targets (Sections 3.4–3.5 of the paper)
  - the 30-day implied volatility index of each index and the 9-day VIX
  - realised volatility of day $t+1$
- Kernels, eq. (3.9) in the paper: $K(\tau) = Z_{\alpha,\delta}^{-\mathbf{1}}(\tau+\delta)^{-\alpha}$, normalised on the 1,000-day window
- Estimation: $(\beta_{\mathbf{0}}, \beta_{\mathbf{1}}, \beta_{\mathbf{2}}, \alpha_{\mathbf{1}}, \delta_{\mathbf{1}}, \alpha_{\mathbf{2}}, \delta_{\mathbf{2}})$ by nonlinear least squares on the training set
  - initialised using a lasso regression on features built with exponential kernels
- Our replication, started from the values in Table 3 of the paper (in parentheses)
  - $\hat{\beta}_{\mathbf{0}} = 0.052$ (0.057), $\hat{\beta}_{\mathbf{1}} = -0.090$ (-0.095)
  - $\hat{\beta}_{\mathbf{2}} = 0.868$ (0.82), $\hat{\alpha}_{\mathbf{1}} = 1.13$ (1.06)
- The 9-day VIX exists only from 2011 (shorter training set)
- The restricted models ($\beta_{\mathbf{1}} = 0$, $\beta_{\mathbf{2}} = 0$) are Seminar 1, C3

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Case Study: FTSE 100, DAX and Further Results of the Paper (1/2)

Index	Implied volatility (paper, Table 2) R^2 train	R^2 test	Realised volatility (paper, Table 4) R^2 train	R^2 test
FTSE 100	0.925	0.870	0.650	0.617
DAX	0.934	0.918	0.722	0.557

- Across all targets $\alpha_2 \in [1.2, 2.3]$ (Tables 3 and 5 of the paper)
  - long memory near criticality (Section 3.2)
- Dropping Σ_t hurts the most
- The model beats the benchmark models (Figures 3.8–3.9 and Appendix C of the paper)
  - two ARCH benchmarks
  - four published path-dependent models

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Case Study: FTSE 100, DAX and Further Results of the Paper (2/2)

- Beyond this course (optional): its Markovian 4-factor version
  - driven by Brownian motion (Section 4.6 of the paper)
  - produces "rough-like" volatility paths, in the words of the authors
  - rough: a path with Hurst exponent $H < 0.5$; Brownian motion has $H = 0.5$
  - no rough volatility is built into the model: the roughness comes from the path dependence

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The Discovery Loop, Reusable Prompts

Literature

- Find peer-reviewed work since 2015 on time variation or structural breaks in tail indices of asset returns, with tests valid under GARCH-type dependence.
- For each: DOI, estimator, test, data.
- Flag entries you are unsure of.

Hypothesis

- State competing hypotheses for a change in Bitcoin's tail after the 2024 spot ETFs.
- What does each imply for the Hill estimate of the tail index of raw and of GARCH-standardised returns?

Data and code

- Use daily Bitcoin closing prices in USD; log returns on its own calendar;
- Hill estimate of the tail index for $|r|$, with k = 2.5% of n, before and after 11 January 2024; moving-block bootstrap standard errors.

Robustness

- Write a pre-registered plan: k from 1% to 5% of n, loss and gain tails, GARCH-standardised residuals, block lengths 10 to 50 days, a placebo date and a placebo asset.

Critique

- Referee this design.
- With 24 tail observations after the break, what is the power against a change of 1 in the tail index?

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Mandatory Checks for the Researcher

- **Wrong return or price series**: reproduce the chapter's Hill table before anything else
- **Choice of k after seeing the result**
  - fix the rule ($k = 2.5\%$ of n) in advance
  - show the Hill plot for 1–5%
- **i.i.d. standard errors**: $\hat{\alpha}/\sqrt{k}$ ignores volatility clustering
  - compare with a block bootstrap
  - for S&P 500, 17 Sep 2014–10 Jan 2024: bootstrap SE 0.817 versus i.i.d. 0.370, hence 2.2 times larger
- **Break date chosen from the data**
  - use a date fixed by the event
  - or a sup-type change-point test with its own critical values Hoga (2017)
- **Bad ticks create fake tails**
  - list the ten largest $|r_t|$ and match each to a dated event
- **Low power**: report the minimum detectable change
  - 2.8 SEs of the difference for a two-sided 5% test with 80% power
  - 2.5 SEs for a one-sided test

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Practical Checklist for Empirical Work

- **Know your data**
  - price vs total-return index
  - adjusted vs raw prices
  - fixing vs traded price
  - official reference rates where possible (EUR/RON)
- **Clean before you model**
  - non-trading days, holidays, stale quotes, obvious errors
  - match the largest moves to real events
- **Use log returns** for single-asset dynamics, simple returns for portfolios
- **Report robust statistics**: quantiles, the number of tail days and the tail index $\hat{\alpha}$ next to kurtosis
- **Annualise carefully**
  - 252 vs 365 periods
  - the $\sqrt{A}$ rule needs uncorrelated returns with constant variance
- **Reproducibility**: code and data on GitHub, as Quantlets

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Further Study on Quantinar

Advanced courses on quantinar.com

- Statistics of Financial Markets
- Tukey's g- and h- transformations
- Applied Time Series Analysis with Python

Code, data and recommended reading

- All charts and tables: github.com/danpele/MFM/tree/main/Quantlets/Ch_01
- Recommended reading: Cont (2001); Franke, Härdle & Hafner (2019); Campbell, Lo & MacKinlay (1997), ch. 1–2

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Acronyms used in this material (1/2)

- **ACF**: AutoCorrelation Function
- **AI**: Artificial Intelligence
- **AR**: AutoRegressive (model)
- **ARCH**: AutoRegressive Conditional Heteroskedasticity
- **ARCH-LM**: ARCH Lagrange Multiplier (test)
- **BET**: Bucharest Exchange Trading (index)
- **BET-TR**: BET Total Return (index)
- **BNR**: Banca Națională a României — *the National Bank of Romania*
- **BVB**: Bursa de Valori București — *the Bucharest Stock Exchange*
- **CAGR**: Compound Annual Growth Rate
- **CI**: Confidence Interval
- **CLT**: Central Limit Theorem
- **CRIX**: CRypto IndeX
- **DAX**: Deutscher Aktienindex — *the German stock index*
- **EGARCH**: Exponential GARCH
- **ETF**: Exchange-Traded Fund
- **FTSE**: Financial Times Stock Exchange (FTSE Russell, index provider)
- **GARCH**: Generalised AutoRegressive Conditional Heteroskedasticity

Acronyms used in this material (2/2)

- **GEO**: Government Emergency Ordinance
- **GJR-GARCH**: Glosten–Jagannathan–Runkle GARCH
- **HAC**: Heteroskedasticity and Autocorrelation Consistent (standard errors)
- **HAR-RV**: Heterogeneous AutoRegressive model of Realised Volatility
- **JB**: Jarque–Bera (test)
- **LB**: Ljung–Box (test)
- **LM**: Lagrange Multiplier
- **MDD**: Maximum DrawDown
- **OHLC**: Open, High, Low, Close
- **PDV**: Path-Dependent Volatility
- **QQ**: Quantile–Quantile (plot)
- **RMSE**: Root Mean Squared Error
- **RS**: Rogers–Satchell (range-based variance estimator)
- **SE**: Standard Error
- **TSPL**: Time-Shifted Power Law (kernel)
- **UK**: United Kingdom
- **US**: United States
- **VaR**: Value-at-Risk
- **VIX**: Cboe Volatility IndeX
- **VIX9D**: Cboe 9-Day Volatility Index
- **YZ**: Yang–Zhang (range-based variance estimator with the overnight jump)

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