



Statistics of Financial Markets

Chapter 14: Fractal Market Hypothesis



Daniel Traian PELE
Stefan Gaman

Bucharest University of Economic Studies
IDA Institute Digital Assets
Blockchain Research Center
AI4EFin Artificial Intelligence for Energy Finance
Romanian Academy, Institute for Economic Forecasting
MSCA Digital Finance

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Motivation

- The **Efficient Market Hypothesis** (EMH) of Fama (1970) has been the dominant paradigm in financial economics for over half a century
- EMH assumes that asset prices fully reflect all available information and that returns follow a random walk with i.i.d. Gaussian innovations:

$$r_t = \mu + \varepsilon_t, \quad \varepsilon_t \stackrel{\text{i.i.d.}}{\sim} \mathcal{N}(0, \sigma^2)$$

- However, EMH **fails to explain** several empirical regularities:
 - ▶ **Fat tails:** $\Pr(|r_t| > x) \gg \Phi(-x)$ for large x , i.e. return distributions exhibit excess kurtosis $\kappa > 3$
 - ▶ **Volatility clustering:** large returns tend to follow large returns, $\text{Corr}(|r_t|, |r_{t+k}|) > 0$ for many lags k
 - ▶ **Long memory:** slow decay of autocorrelations, $\rho(k) \sim Ck^{2H-2}$ as $k \rightarrow \infty$ for $H > 0.5$
 - ▶ **Market crashes:** sudden, correlated drops that are orders of magnitude beyond Gaussian predictions (e.g., Black Monday 1987: -22.6%)
- The **Fractal Market Hypothesis** (FMH), proposed by Peters (1991, 1994), offers an alternative framework based on fractal geometry, heterogeneous time horizons, and the concept of market liquidity

EMH vs FMH

Efficient Market Hypothesis (EMH)

- All investors have **the same investment horizon** and react identically to information
- Information is **immediately and fully** reflected in asset prices
- Returns are **i.i.d.** with a well-defined variance:

$$r_t \stackrel{\text{i.i.d.}}{\sim} F(\mu, \sigma^2)$$

- Markets are always in **equilibrium**; arbitrage opportunities vanish instantaneously
- Price changes follow a **random walk** $\succ$ no serial dependence
- Risk is fully captured by **variance** σ^2 (mean-variance framework)

Fractal Market Hypothesis (FMH)

- Investors operate on **multiple time horizons**: from intraday to multi-year
- Information is **valued differently** depending on the investor's horizon
- Market **stability** depends on the **diversity of horizons**:

$$\text{Liquidity} \propto \text{Horizon diversity}$$

- When horizons **collapse** (all investors panic simultaneously), liquidity evaporates and crashes occur
- Returns exhibit **self-similarity** and **long-range dependence**

- Risk is characterized by the **Hurst exponent** H and the full distributional shape

Self-Similarity

Definition 1 (Statistical Self-Similarity)

A stochastic process $\{X(t)\}_{t \geq 0}$ is called **self-similar** with index $H > 0$ (or H -ss) if for every $c > 0$:

$$\{X(ct)\}_{t \geq 0} \stackrel{d}{=} \{c^H X(t)\}_{t \geq 0}$$

where $\stackrel{d}{=}$ denotes equality of all finite-dimensional distributions.

- The parameter H is called the **self-similarity exponent** (or Hurst exponent)
- **Implications for returns:** if the log-price process $\ln P(t)$ is H -self-similar, then rescaling time by c is equivalent to rescaling returns by c^H :

$$r(t, \Delta) = \ln P(t + \Delta) - \ln P(t) \stackrel{d}{=} c^H r\left(\frac{t}{c}, \frac{\Delta}{c}\right)$$

- For returns over horizon Δ , the **volatility scaling law** follows:

$$\sigma(\Delta) = \sigma(1) \cdot \Delta^H$$

- Under EMH ($H = 0.5$), this gives the classical square-root-of-time rule: $\sigma(\Delta) = \sigma(1)\sqrt{\Delta}$
- For $H \neq 0.5$, the scaling deviates $\succ$ evidence **against** EMH

Fractional Brownian Motion

Definition 2 (Fractional Brownian Motion)

A **fractional Brownian motion** (fBm) $\{B_H(t)\}_{t \in \mathbb{R}}$ with Hurst parameter $H \in (0, 1)$ is the unique centered Gaussian process with $B_H(0) = 0$ and covariance:

$$\mathbb{E}[B_H(t) B_H(s)] = \frac{1}{2} (|t|^{2H} + |s|^{2H} - |t - s|^{2H})$$

□ Key properties of fBm:

- ▶ Self-similar with exponent H : $B_H(ct) \stackrel{d}{=} c^H B_H(t)$
- ▶ Stationary increments: $B_H(t + s) - B_H(s) \stackrel{d}{=} B_H(t)$
- ▶ Variance of increments: $\mathbb{E}[(B_H(t) - B_H(s))^2] = |t - s|^{2H}$

□ Three regimes based on H :

- ▶ $H = 0.5$: **Standard Brownian motion** $\succ$ independent increments, no memory
- ▶ $H > 0.5$: **Persistent** (trending) $\succ$ positive autocorrelation of increments,
 $\text{Corr}(\Delta B_H(t), \Delta B_H(t + k)) > 0$
- ▶ $H < 0.5$: **Anti-persistent** (mean-reverting) $\succ$ negative autocorrelation,
 $\text{Corr}(\Delta B_H(t), \Delta B_H(t + k)) < 0$

- The autocovariance of increments at lag k is:

Long Memory Processes

Definition 3 (ARFIMA Process)

A process $\{y_t\}$ follows an **ARFIMA**(p, d, q) model if:

$$\Phi(L)(1-L)^d y_t = \Theta(L)\varepsilon_t, \quad \varepsilon_t \stackrel{\text{i.i.d.}}{\sim} (0, \sigma_\varepsilon^2)$$

where $d \in (-0.5, 0.5)$ is the fractional differencing parameter, $\Phi(L) = 1 - \phi_1 L - \dots - \phi_p L^p$ is the AR polynomial, and $\Theta(L) = 1 + \theta_1 L + \dots + \theta_q L^q$ is the MA polynomial.

- The **fractional difference operator** $(1-L)^d$ is defined via its binomial expansion:

$$(1-L)^d = \sum_{k=0}^{\infty} \binom{d}{k} (-L)^k = 1 - dL - \frac{d(1-d)}{2!} L^2 - \frac{d(1-d)(2-d)}{3!} L^3 - \dots$$

- **Connection to the Hurst exponent:** for a pure ARFIMA(0, d , 0) process,

$$d = H - \frac{1}{2}$$

- The spectral density near frequency zero behaves as $f(\lambda) \sim C_f |\lambda|^{-2d}$ as $\lambda \rightarrow 0$

- **Long memory** ($0 < d < 0.5$, equivalently $H > 0.5$): the autocorrelation function decays hyperbolically, $\rho(k) \sim C k^{2d-1}$ as $k \rightarrow \infty$, and $\sum_{k=-\infty}^{\infty} |\rho(k)| = \infty$

Hurst Exponent: Definition

- ▣ Named after **Harold Edwin Hurst** (1880–1978), a British hydrologist who studied the long-term storage capacity of the Nile River
- ▣ Hurst (1951) found that the annual minima and maxima of the Nile's water levels exhibited **long-range dependence**: wet years tended to cluster, as did dry years
- ▣ He discovered that the rescaled range R/S scaled as n^H with $H \approx 0.73$, far exceeding $H = 0.5$ expected for independent observations

Definition 4 (Hurst Exponent)

The **Hurst exponent** $H \in (0, 1)$ characterizes the long-range dependence structure of a time series. Its value determines the nature of the process:

- ▣ $H = 0.5$: **Random walk** (no memory) $\succ$ independent increments, consistent with EMH
- ▣ $H > 0.5$: **Persistent** / trending behavior $\succ$ positive long-range autocorrelation; past trends tend to continue
- ▣ $H < 0.5$: **Anti-persistent** / mean-reverting $\succ$ negative long-range autocorrelation; past trends tend to reverse
- ▣ For a self-similar process, the variance of aggregated returns over a window of size n scales as:

R/S Analysis

- The **Rescaled Range** (R/S) analysis is the classical method for estimating H (Hurst, 1951; Mandelbrot & Wallis, 1969)
- **Algorithm:** given a time series $\{x_1, x_2, \dots, x_N\}$:
 1. Divide the series into blocks of size n
 2. For each block $\{x_1, \dots, x_n\}$, compute the mean $\bar{x}_n = \frac{1}{n} \sum_{i=1}^n x_i$
 3. Compute the **cumulative deviations**: $Y_k = \sum_{i=1}^k (x_i - \bar{x}_n)$, $k = 1, \dots, n$
 4. Compute the **range**: $R(n) = \max_{1 \leq k \leq n} Y_k - \min_{1 \leq k \leq n} Y_k$
 5. Compute the **standard deviation**: $S(n) = \sqrt{\frac{1}{n} \sum_{i=1}^n (x_i - \bar{x}_n)^2}$
 6. Compute the **rescaled range**: $(R/S)_n = R(n)/S(n)$
- The **Hurst law**: as $n \rightarrow \infty$,

$$\mathbb{E} \left[\frac{R(n)}{S(n)} \right] \propto n^H$$

- Estimate H by **OLS regression** of $\log(R/S)$ on $\log(n)$:

$$\log \left(\frac{R(n)}{S(n)} \right) = \hat{H} \cdot \log(n) + c + \text{error}$$

- **Limitation:** R/S analysis is sensitive to short-range dependence and non-stationarity; Lo (1991) proposed a modified R/S statistic that accounts for short-range correlations

Detrended Fluctuation Analysis (DFA)

- **DFA** was introduced by Peng et al. (1994) as a robust alternative to R/S analysis, especially for non-stationary time series
- **Algorithm:** given a time series $\{x_1, \dots, x_N\}$:
 1. Compute the **integrated (profile) series**: $Y_k = \sum_{i=1}^k (x_i - \bar{x})$, $k = 1, \dots, N$
 2. Divide $\{Y_k\}$ into non-overlapping segments of length n
 3. In each segment, fit a polynomial trend $\hat{Y}_k^{(m)}$ of order m (DFA- m ; linear: $m = 1$, quadratic: $m = 2$)
 4. Compute the **fluctuation function**:

$$F(n) = \sqrt{\frac{1}{N} \sum_{k=1}^N \left(Y_k - \hat{Y}_k^{(m)} \right)^2}$$

5. The DFA scaling law gives: $F(n) \propto n^H$
- Estimate H from the slope of $\log F(n)$ vs. $\log n$
 - **Advantages over R/S analysis:**
 - ▶ Robust to **polynomial trends** (detrending removes non-stationarity artifacts)
 - ▶ Works for **non-stationary** series (integrated processes)
 - ▶ Higher-order DFA ($m \geq 2$) removes more complex trends
 - **Interpretation of DFA exponents:** $H < 0.5$: anti-persistent; $H = 0.5$: uncorrelated; $0.5 < H < 1$: long-range persistent; $H = 1$: $1/f$ noise; $H > 1$: non-stationary, unbounded

Multiple Investment Horizons

- The **central insight** of the Fractal Market Hypothesis: financial markets are populated by investors operating on **vastly different time scales**:
 - ▶ **High-frequency traders**: holding periods of milliseconds to minutes
 - ▶ **Day traders**: intraday to several days
 - ▶ **Swing/position traders**: weeks to months
 - ▶ **Mutual funds and hedge funds**: months to years
 - ▶ **Pension funds and sovereign wealth funds**: years to decades
- Each class of investor **values information differently** depending on their horizon:
 - ▶ A day trader reacts strongly to a disappointing earnings report
 - ▶ A pension fund ignores short-term noise and focuses on long-term fundamentals
- **Market stability** requires sufficient **diversity of horizons**. When short-horizon sellers panic, long-horizon investors step in to buy, providing liquidity:

Stable market $\iff$ Broad distribution of investment horizons

- The **fractal structure** of the market arises because the same price patterns appear at different time scales $\succ$ statistical self-similarity of returns
- Peters (1994) argues that a well-functioning market is like a fractal: **remove any one scale and the structure breaks down**

Liquidity and Market Crashes

- Under FMH, **liquidity** is not an abstract concept but a direct consequence of horizon diversity:

$$\text{Liquidity}_t \propto \sum_{h \in \mathcal{H}} w_h \cdot \text{Activity}_{h,t}$$

where $\mathcal{H}$ is the set of investment horizons and w_h are horizon-specific weights

- Normal market conditions:** investors at different horizons provide a **continuous flow** of buy and sell orders $\succ$ tight bid-ask spreads, deep order books
- Crisis mechanism** (FMH explanation of crashes):
 - A **fundamental shock** (e.g., Lehman collapse, COVID-19 pandemic) affects **all horizons simultaneously**
 - Long-horizon investors abandon their usual strategies and adopt **short-term behavior** (panic selling)
 - The distribution of horizons **collapses** $\succ$ all investors effectively have the same horizon
 - With no diversity, **liquidity evaporates**: there are no buyers for the sellers
 - The **fractal structure breaks down** $\succ$ the Hurst exponent shifts abruptly
- Empirical evidence:**
 - 2008 GFC: the Hurst exponent of the S&P 500 dropped sharply, indicating a loss of fractal structure (Dominique & Rivera, 2011)
 - COVID-19 (March 2020): unprecedented volatility and illiquidity as horizon diversity collapsed within

Hurst Exponent Estimates

- Extensive empirical literature documents departures from $H = 0.5$ across asset classes

Asset class	Market	$\hat{H}$ (typical)	Method
Stock indices	S&P 500	0.55–0.62	R/S, DFA
Stock indices	FTSE 100	0.53–0.60	DFA
Stock indices	DAX	0.54–0.61	R/S, DFA
Emerging markets	BIST 100, BET	0.58–0.68	DFA
Cryptocurrencies	Bitcoin	0.50–0.75	DFA, wavelet
Cryptocurrencies	Ethereum	0.52–0.70	DFA
Forex	EUR/USD	0.50–0.55	R/S, DFA
Commodities	Gold	0.52–0.58	DFA
Bonds	US 10Y Treasury	0.55–0.65	R/S

- Key findings:** most equity indices exhibit mild persistence ($H \approx 0.55\text{--}0.65$), consistent with FMH
- Cryptocurrencies show **highly variable** H , reflecting evolving market microstructure and varying degrees of efficiency

Multifractal Models

- A single Hurst exponent H (monofractal) may be insufficient to capture the full complexity of financial returns
- Mandelbrot, Fisher, and Calvet (1997) introduced the **Multifractal Model of Asset Returns** (MMAR)
- In the MMAR, the log-price is modeled as:

$$\ln P(t) = B_H[\theta(t)]$$

where B_H is a fractional Brownian motion and $\theta(t)$ is a **multifractal trading time** (a random measure)

- The process is **multifractal** if its q -th absolute moments scale as:

$$\mathbb{E}[|X(t)|^q] = c(q) \cdot t^{\tau(q)+1}$$

where $\tau(q)$ is a **nonlinear** function of q (for monofractal processes, $\tau(q) = qH - 1$ is linear)

- The **multifractal spectrum** $f(\alpha)$ describes the distribution of local Hölder exponents α :

$$f(\alpha) = \inf_q \{q\alpha - \tau(q)\} \quad (\text{Legendre transform})$$

- A **wide** $f(\alpha)$ spectrum indicates strong multifractality $\succ$ the series has regions of very different local scaling behavior
- Advantages over monofractal models: captures **volatility clustering**, **fat tails**, and **intermittency** simultaneously

Applications

- **Estimating the Hurst exponent in Python:** use the `nolds` or `hurst` libraries for R/S and DFA:

```
import numpy as np
import nolds

# Simulated fractional Brownian motion (H = 0.7)
from fbm import FBM
f = FBM(n=10000, hurst=0.7, length=1, method='daviesharte')
fbm_sample = f.fbm()
increments = np.diff(fbm_sample)

# R/S estimate
H_rs = nolds.hurst_rs(increments)
print(f"R/S_Hurst_estimate: {H_rs:.4f}")

# DFA estimate
H_dfa = nolds.dfa(increments)
print(f"DFA_Hurst_estimate: {H_dfa:.4f}")
```

- **Rolling Hurst exponent:** compute $\hat{H}_t$ in a rolling window of size w to detect **regime changes**:

$$\hat{H}_t = \text{DFA}(\{r_{t-w+1}, \dots, r_t\}), \quad t = w, w+1, \dots, T$$

- A **sudden drop** in $\hat{H}_t$ towards 0.5 (or below) signals loss of fractal structure $\succ$ potential crisis indicator
- **Practical application:** FMH-based risk management monitors horizon diversity via time-varying H_t estimates as an **early warning signal** for market stress

Conclusions

- The **Fractal Market Hypothesis** provides a richer framework than EMH for understanding financial markets:
 - ▶ Markets are stable when populated by investors with **diverse time horizons**
 - ▶ Crashes occur when this diversity **collapses** and all investors act on the same short-term horizon
 - ▶ Returns exhibit **self-similarity** and **long-range dependence**, captured by the Hurst exponent H
- **Comparison with EMH:**
 - ▶ EMH: $H = 0.5$, i.i.d. returns, no crashes beyond Gaussian tails
 - ▶ FMH: $H \neq 0.5$, fractal scaling, horizon-dependent liquidity, structural explanation for crashes
- **Practical implications for risk management:**
 - ▶ Standard VaR models (assuming $H = 0.5$) **underestimate tail risk** when $H > 0.5$
 - ▶ Volatility scaling: $\sigma(\Delta) = \sigma(1) \cdot \Delta^H$ instead of $\sigma(1) \cdot \Delta^{0.5}$
 - ▶ Monitoring the time-varying Hurst exponent $\hat{H}_t$ as an **early warning indicator** for liquidity crises
 - ▶ Multifractal models (MMAR) offer improved density forecasts for portfolio optimization
- **Open research directions:** connection between FMH and adaptive market hypothesis (Lo, 2004), high-frequency fractal analysis, multifractal portfolio theory

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