

Statistics of Financial Markets

Chapter 8: Volatility estimators

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Acronyms used in this material (1/2)

- **ACF**: Autocorrelation Function
- **AI**: Artificial Intelligence
- **AR**: AutoRegressive (model); Abnormal Return
- **ARCH**: AutoRegressive Conditional Heteroskedasticity
- **ARCH-LM**: ARCH Lagrange Multiplier (test)
- **BET**: Bucharest Exchange Trading (index)
- **BVB**: Bursa de Valori București (the Bucharest Stock Exchange)
- **CBOE**: Chicago Board Options Exchange
- **DM**: Diebold–Mariano (test)
- **EODHD**: EOD Historical Data (market-data provider)
- **ES**: Expected Shortfall
- **EWMA**: Exponentially Weighted Moving Average
- **GARCH**: Generalised AutoRegressive Conditional Heteroskedasticity

Acronyms used in this material (2/2)

- **GK**: Garman–Klass (volatility estimator)
- **HAR**: Heterogeneous AutoRegressive (model)
- **IGARCH**: Integrated GARCH
- **LM**: Lagrange Multiplier
- **MSE**: Mean Squared Error
- **MZ**: Mincer–Zarnowitz (regression)
- **OHLC**: Open, High, Low, Close
- **OMV**: Österreichische Mineralölverwaltung — Austrian Mineral Oil Administration (OMV group)
- **QLIKE**: Quasi-LIKElihood (loss function)
- **RV**: Realised Variance
- **SE**: Standard Error
- **SFE**: Statistics of Financial Markets, Exercises (Quantlet collection of the textbook)
- **VIX**: CBOE Volatility IndeX
- **YZ**: Yang–Zhang (volatility estimator)

Today's Question and Route

- **Question**: how large is the risk of an asset today, if we can never observe it directly?
  - Chapter 7: returns are almost unpredictable, but their size is not; here we measure that size
  - Chapter 9 turns the measurements into a model (ARCH and GARCH)
- **Route** of the chapter
  - what volatility is and why it is latent; historical volatility and the choice of the window
  - EWMA (RiskMetrics); range-based estimators from open, high, low and close prices
  - realised volatility; volatility clustering and its tests
  - long-run behaviour, the VIX, the leverage effect; how to compare volatility forecasts

Learning Outcomes

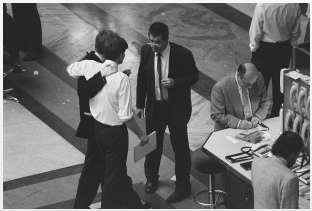
- Define volatility, annualise it with the actual trading frequency and explain why it is a latent quantity
- Compute historical and EWMA volatility and choose a window or a decay factor
- Compute the Parkinson, Garman–Klass, Rogers–Satchell and Yang–Zhang estimators and state their assumptions
- Detect volatility clustering with the ACF of squared returns, the Ljung–Box test and the ARCH-LM test
- Compare implied and realised volatility, and evaluate volatility forecasts with MSE and QLIKE

Reading and Tools

- Textbook: Franke, Härdle & Hafner (2019), Ch. 13 (introduction: volatility and conditional heteroskedasticity)
  - exercises with solutions: Borak, Härdle & López-Cabrera (2013), Ch. 13
- Volatility and its stylised facts: Tsay (2010), Ch. 3.1; a review of volatility forecasting: Poon & Granger (2003)
- Python Quantlets of this chapter: Quantlets/Ch_08
  - ported from the SFE Quantlets SFEkurgarch (kurtosis of a GARCH process) and SFEvolnonparest (nonparametric volatility)
- Lecture notebook: open in Google Colab
- Video course: Applied Time Series Analysis with Python

Volatility Clustering and the Uses of Volatility

- "Large changes tend to be followed by large changes, of either sign, and small changes tend to be followed by small changes" Mandelbrot (1963)
  - after the crash of 19 October 1987, markets stayed turbulent for weeks
- Volatility enters almost every number in finance
  - risk measures: VaR and ES (Chapter 10), the Sharpe ratio (Chapter 1)
  - option prices, margins of exchanges, the size of positions
- This chapter: how to **measure** it from prices, and how to tell whether it clusters



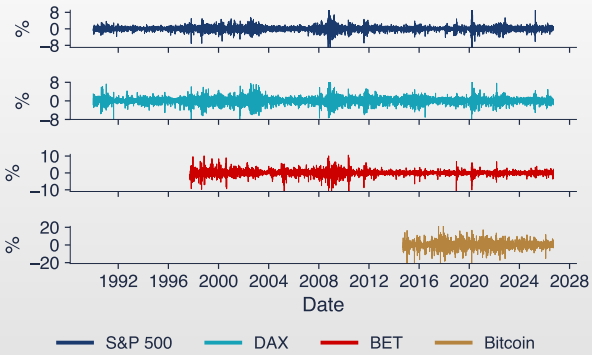
Amsterdam Stock Exchange, 21 October 1987, two days after the crash

Photo: Bart Molendijk / Anefo (1987); CC0; Wikimedia Commons

Volatility: Definition and Units

- Daily log return in %: $r_t = 100\,(\ln P_t - \ln P_{t-1})$ (Chapter 1)
  - **volatility** = the standard deviation of returns, $\sigma = \sqrt{\mathrm{Var}(r_t)}$
  - it has the units of the return (% per day); the variance σ^2 has units of $\%^2$
- **Annualisation**: with A observations per year and uncorrelated returns, $\sigma_{\text{year}} = \sigma_{\text{day}}\sqrt{A}$
  - A = the actual frequency: about 252 for the S&P 500, 250 for the BET, 365 for Bitcoin (trades every day)
  - the rule $\sqrt{A}$ is the square-root-of-time rule of Chapter 7: it needs uncorrelated returns
- Volatility measures dispersion in both directions: a large gain counts as much as a large loss
  - other measures of uncertainty exist, for example the information entropy of returns Pele, Lazar & Dufour (2017)

Volatility Clustering in Four Markets



- Daily log returns in %: S&P 500 and DAX since 1990, BET since 1997, Bitcoin since 2014; last day 18 September 2026
- Quiet years alternate with turbulent months in all four series

SFM_ch8_returns_volatility

Interpretation: Volatility in Four Markets

	Annual volatility	Largest daily move	Date	Excess kurtosis
S&P 500	18.0%	−12.8%	16 March 2020	10.9
DAX	21.8%	−13.1%	12 March 2020	6.0
BET	23.4%	−13.1%	7 January 2009	9.9
Bitcoin	66.6%	−46.5%	12 March 2020	12.0

- The largest moves are 8 to 13 standard deviations: practically impossible under the Normal distribution (Chapter 2)
- Excess kurtosis far above 0 (the Normal value): heavy tails; Section 6 shows that clustering is one cause

SFM_ch8_returns_volatility

The Latent Nature of Volatility

- Model: the return is a constant mean plus a shock whose size changes every day: $r_t = \mu + \sigma_t z_t$
  - μ: the mean return; z_t: i.i.d. shocks with mean 0 and variance 1
  - σ_t is the **conditional volatility**: $\sigma_t^2 = \mathrm{Var}(r_t \mid \mathcal{F}_{t-1})$, with $\mathcal{F}_{t-1}$ the information at the close of day $t-1$
  - the **unconditional volatility** $\sigma = \sqrt{\mathrm{Var}(r_t)}$ is its long-run average level
- **Latent**: we never observe σ_t, only one return per day
  - r_t^2 is an unbiased but very noisy measure of σ_t^2 (with $\mu = 0$): $E[r_t^2 \mid \mathcal{F}_{t-1}] = \sigma_t^2$
  - a quiet day can happen in a turbulent period, and the other way round
- Three ways to get more information about σ_t
  - average many days: historical volatility, EWMA
  - use the whole path of the day: the high and low prices
  - use intraday prices: realised volatility

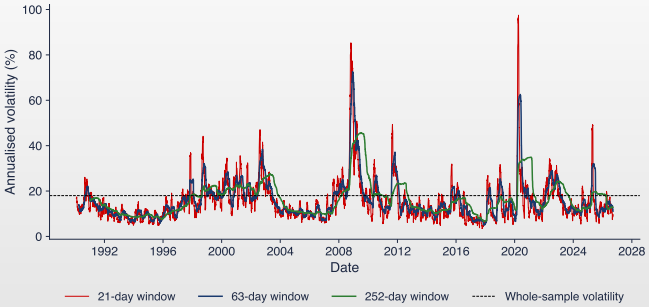
Worked Example: Annualising Volatility

- S&P 500 since 1990: daily standard deviation $\hat{\sigma} = 1.1337\%$, $A = 251.7$ observations per year (the actual frequency)
  - $\hat{\sigma}_{\text{year}} = 1.1337 \times \sqrt{251.7} = 1.1337 \times 15.866 = 18.0\%$
- Bitcoin since 2014: $\hat{\sigma} = 3.4852\%$ per day, 7 days a week
  - correct: $3.4852 \times \sqrt{365.3} = 3.4852 \times 19.114 = 66.6\%$
  - wrong, with the equity convention: $3.4852 \times \sqrt{252} = 3.4852 \times 15.875 = 55.3\%$
- Comparing assets: annualise each with its own calendar, then compare

Historical (Rolling) Volatility

- **Historical volatility** on a window of the last n days:
  - $\hat{\sigma}_t = \sqrt{\frac{1}{n-1}\sum_{j=0}^{n-1}(r_{t-j} - \bar{r}_t)^2}$, with $\bar{r}_t$ the mean of the same n returns
  - j: how many days back the return lies ($j = 0$ is day t); $\hat{\sigma}_t$: the estimate at the close of day t
  - **rolling window**: each day the newest return enters and the oldest leaves
- Zero-mean version: $\hat{\sigma}_t^2 = \frac{1}{n}\sum_{j=0}^{n-1} r_{t-j}^2$
  - for daily data the mean is tiny compared with σ (0.03% against 1.13% for the S&P 500); setting it to 0 removes a noisy estimate
- Usual windows: $n = 21$ (a month), 63 (a quarter), 252 (a year) of trading days
  - each $\hat{\sigma}_t$ is annualised with $\sqrt{A}$ and plotted at the last day of its window

S&P 500: Rolling Volatility with 21-, 63- and 252-Day Windows



- Rolling volatility of the S&P 500, annualised; dashed line: the whole-sample volatility, 18.0%
- **Question for the room**: which window would you trust on the day after a crash?

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Interpretation: the Window Trade-Off

- Short window ($n = 21$): reacts fast, but is noisy
  - maximum 97.5% on 6 April 2020; the typical daily change of $\ln \hat{\sigma}_t$ is 6.7%
- Long window ($n = 252$): smooth, but slow
  - maximum only 45.6%, reached on 15 July 2009, months after the peak of the 2008 crisis; daily change 0.8%
- **Answer**: a short window, or EWMA (next section), because a long window still averages the calm days before the crash
  - for capital planning, a long window is preferred: it changes slowly
- At the end of the sample: 9.6% (21 days), 11.1% (63 days), 12.9% (252 days)

SFM_ch8_returns_volatility

Precision of a Sample Volatility

- With n i.i.d. Normal returns, the relative standard error of $\hat{\sigma}$ is $\mathrm{SE}(\hat{\sigma})/\sigma \approx 1/\sqrt{2n}$
  - $\mathrm{SE}(\hat{\sigma})$: the standard deviation of the estimate across samples; divided by σ it is a relative error
  - $n = 21$: 15.4%; $n = 63$: 8.9%; $n = 252$: 4.5%
- With heavy tails (kurtosis κ): $\mathrm{SE}(\hat{\sigma})/\sigma \approx \sqrt{(\kappa - 1)/(4n)}$ (delta method)
  - S&P 500: $\kappa = 13.9$, so $n = 21$: 39%; $n = 252$: 11%
  - for $\kappa = 3$ (Normal) the two formulas coincide
- **Worked example**: the last 21-day volatility of the S&P 500 is 9.6%
  - approximate 95% interval, Normal returns: [6.7%; 12.5%]; with the observed kurtosis: [2.2%; 17.0%]

1994: RiskMetrics and Daily Volatility Estimates

- J.P. Morgan launched RiskMetrics in October 1994 and published its methods and data free of charge J.P. Morgan/Reuters (1996)
  - goal: one common yardstick for market risk across banks
  - volatilities and correlations of more than 480 series, updated every day
- The method: an exponentially weighted moving average (EWMA) of squared returns
  - one parameter, the **decay factor** λ: 0.94 for daily forecasts, 0.97 for monthly ones
  - for each series, λ minimises the root mean squared error of its variance forecasts; 0.94 is the weighted average over all series



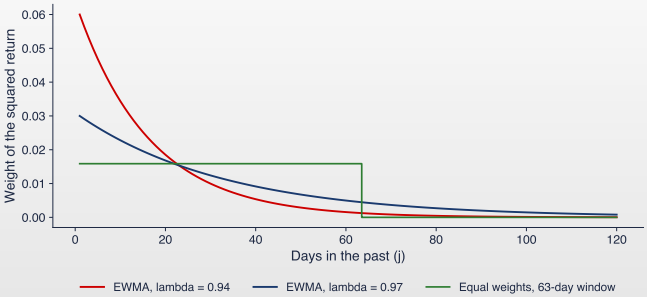
The J.P. Morgan building at 23 Wall Street, New York

Photo: Beyond My Ken (2012); CC BY-SA 4.0; Wikimedia Commons

The EWMA Estimator

- **EWMA** recursion (zero mean): $\sigma_t^2 = \lambda\,\sigma_{t-1}^2 + (1-\lambda)\,r_{t-1}^2,\ \ 0 < \lambda < 1$
  - today's variance = a weighted average of yesterday's variance and yesterday's squared return
  - λ: the weight of the past; a λ close to 1 gives a smooth, slowly reacting estimate
  - σ_t^2 uses returns up to day $t-1$: it is the forecast for day t
  - start: σ_1^2 = the mean of the first squared returns; its effect fades after a few months
- Unrolling the recursion: $\sigma_t^2 = (1-\lambda)\sum_{j\geq 1}\lambda^{j-1} r_{t-j}^2$
  - weights $(1-\lambda)\lambda^{j-1}$ decline geometrically and sum to 1
  - **half-life**: the lag at which the weight halves, $\ln 0.5 / \ln \lambda$
- A rolling window gives weight $1/n$ to each of the last n days and 0 to all older days

EWMA Weights against a Rolling Window



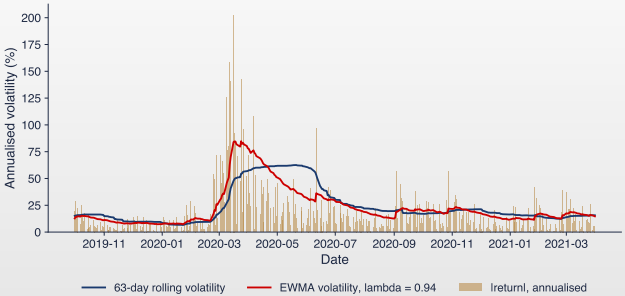
- $\lambda = 0.94$: half-life 11.2 days, mean lag $1/(1-\lambda) = 16.7$ days, 99% of the weight in the last 75 days
- $\lambda = 0.97$: half-life 22.8 days, 99% of the weight in the last 152 days

SFM_ch8_ewma

Worked Example: One EWMA Update

- Yesterday's EWMA variance: $\sigma^2_{t-1} = 1.00$ (%^{2}), a daily volatility of 1%; yesterday's return: $r_{t-1} = -3\%$
  - step 1: $\lambda\sigma^2_{t-1} = 0.94 \times 1.00 = 0.94$
  - step 2: $(1-\lambda)r^2_{t-1} = 0.06 \times 9 = 0.54$
  - step 3: $\sigma^2_t = 0.94 + 0.54 = 1.48$, so $\sigma_t = 1.217\%$ per day, $1.217 \times \sqrt{252} = 19.3\%$ per year
- If today is quiet ($r_t = 0$): $\sigma^2_{t+1} = 0.94 \times 1.48 = 1.39$
  - the shock fades by 6% per day; it never drops out abruptly
- A 63-day window would add $9/63 = 0.14$ to the variance today and remove this contribution at once, 63 days later

The Ghost Effect: March–June 2020



- S&P 500: 63-day rolling volatility and EWMA volatility ($\lambda = 0.94$), annualised; bars: $|r_t|$ on the same annual scale
- **Question for the room**: why does the blue line fall sharply in mid-June 2020, on a calm day?

SFM_ch8_ewma

Interpretation: the Ghost Effect and EWMA Weighting

- **Answer**: on 15 June 2020 the return of 16 March 2020 (-12.8%) left the 63-day window
  - the rolling volatility fell from 50.4% to 43.0% in one day, with no news at all
  - this artefact is the **ghost effect**: an old shock leaves the window abruptly
- EWMA reacts faster and forgets smoothly
  - EWMA peak: 84.8% on 24 March 2020; 63-day peak: 62.6% only on 20 May 2020
  - on the ghost day EWMA changed by only -0.9 percentage points
- Price of the speed: EWMA is noisier than a long window and has no long-run level to return to

SFM_ch8_ewma

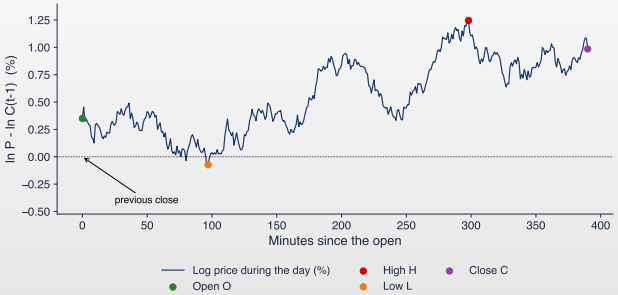
EWMA and GARCH: a Pointer to Chapter 9

- **ARCH** Engle (1982) and **GARCH** Bollerslev (1986): $\sigma_t^2 = \omega + \alpha\, r_{t-1}^2 + \beta\, \sigma_{t-1}^2$
  - $\omega > 0$: a constant; α: the reaction to yesterday's squared return; β: the weight of yesterday's variance
  - EWMA is the special case $\omega = 0$, $\alpha = 1 - \lambda$, $\beta = \lambda$, so $\alpha + \beta = 1$ ("integrated" GARCH, **IGARCH**)
- Consequences for forecasts made at day t for day $t + h$
  - EWMA: flat, $E_t[\sigma_{t+h}^2] = \sigma_{t+1}^2$ for every h (E_t: expectation given the information of day t): today's level persists for ever
  - GARCH with $\alpha + \beta < 1$: pulled towards the long-run variance $\omega/(1 - \alpha - \beta)$
- Chapter 9 estimates ω, α, β by maximum likelihood; here λ is fixed or chosen by a forecast criterion (Section 8)

Open, High, Low and Close Prices

- **OHLC** data: for each day t, the open O_t, the high H_t, the low L_t and the close C_t
  - the close-to-close return uses 2 prices a day; H_t and L_t carry information about the whole path of the day
- Notation (log differences in %), each from the open of day t:
  - $u_t = \ln(H_t/O_t)$, $d_t = \ln(L_t/O_t)$, $c_t = \ln(C_t/O_t)$; so $d_t \leq 0 \leq u_t$ and $d_t \leq c_t \leq u_t$
  - **overnight** return $o_t = \ln(O_t/C_{t-1})$; the daily return is $r_t = o_t + c_t$
  - the **range** $u_t - d_t = \ln(H_t/L_t)$
- Data: S&P 500 since 2008, DAX since 2006, Bitcoin since 2014, BVB stocks since 2010; the BET file has closes only

One Trading Day in OHLC Data



- Simulated log price (Brownian motion, 390 one-minute steps) after an overnight gap of 0.35%: the close keeps only the end point; high and low summarise the path
- Two days with the same close can have very different ranges, and so very different risk

SFM_ch8_range_estimators

The Parkinson Estimator (1980)

- For a Brownian motion with no drift and variance σ^2 per day: $E[(u_t - d_t)^2] = 4 \ln 2 \, \sigma^2$ Parkinson (1980)
  - in words: the squared log range, rescaled so that on average it equals the variance
  - one day: $\hat{\sigma}^2_{P,t} = \dfrac{(u_t - d_t)^2}{4 \ln 2}$; on n days: the mean of these values
  - $4 \ln 2 \approx 2.77$: the range is on average larger than σ
- Assumptions
  - continuous trading during the day, so the observed high and low are the true extremes
  - no drift and no overnight jump: the range only sees the trading session
- Much more precise than r_t^2: a simulation later in this section measures by how much
  - the log range is also used to estimate stochastic volatility models Alizadeh, Brandt & Diebold (2002)

Garman–Klass (1980) and Rogers–Satchell (1991)

- **Garman–Klass** Garman & Klass (1980): adds the open-to-close return
  - $\hat{\sigma}^2_{GK,t} = 0.5\,(u_t - d_t)^2 - (2\ln 2 - 1)\,c_t^2$
  - the weights make the estimator unbiased and nearly of minimum variance among such quadratic forms, under no drift
- **Rogers–Satchell** Rogers & Satchell (1991): unbiased for any drift μ
  - $\hat{\sigma}^2_{RS,t} = u_t(u_t - c_t) + d_t(d_t - c_t)$
  - useful for assets with a strong trend over the window (for example, Bitcoin in a strong rally)
- Both still ignore the overnight return o_t: they measure the variance of the trading session only

Yang–Zhang (2000): Including the Overnight Return

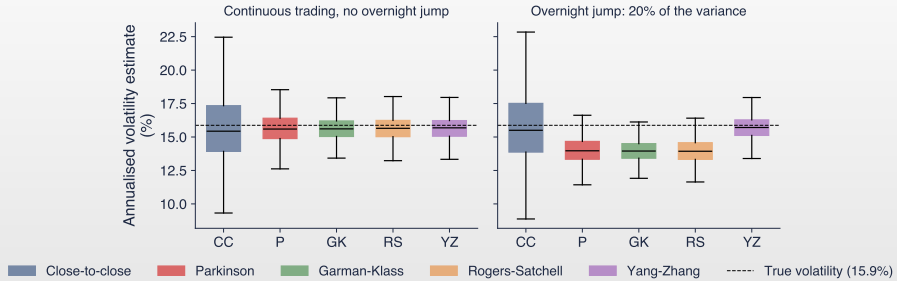
- On a window of n days Yang & Zhang (2000):
  - $\hat{\sigma}^2_{YZ} = \hat{\sigma}^2_O + k\,\hat{\sigma}^2_C + (1-k)\,\hat{\sigma}^2_{RS}, \quad k = \dfrac{0.34}{1.34 + (n+1)/(n-1)}$
  - $\hat{\sigma}^2_O$, $\hat{\sigma}^2_C$: sample variances of the overnight returns o_t and of the open-to-close returns c_t; $\hat{\sigma}^2_{RS}$: the mean Rogers–Satchell term
- Properties
  - unbiased with drift and with opening jumps; k minimises the variance of the estimator
  - $n = 21$: $k = 0.34/(1.34 + 22/20) = 0.34/2.44 = 0.14$
- Needs a window of $n \geq 2$ days (sample variances); Parkinson, Garman–Klass and Rogers–Satchell also work for a single day

Worked Example: S&P 500 on 16 March 2020

- Previous close 2 711; open 2 509, high 2 563, low 2 381, close 2 386
  - $o = -7.760$, $u = 2.145$, $d = -5.223$, $c = -5.005$ (in %); close-to-close $r = o + c = -12.765$
- One-day variance estimates ($\%^2$)
  - close-to-close: $r^2 = 163.0$
  - Parkinson: $(7.368)^2/2.773 = 19.6$; Garman–Klass: 17.5; Rogers–Satchell: 16.5
  - overnight part $o^2 = 60.2$; overnight plus Rogers–Satchell: 76.7
- Most of the fall happened overnight, before the open: estimators that ignore o_t miss most of the risk of this day

SFM_ch8_range_estimators

Efficiency by Simulation



- 1000 windows of 21 days; true volatility 1% per day (15.9% per year); intraday Brownian motion on 4680 steps (one every 5 seconds)
- **Efficiency** of an estimator: $\mathrm{Var}(\hat{\sigma}^2_{CC})/\mathrm{Var}(\hat{\sigma}^2)$, with CC the close-to-close estimator; right panel: 20% of the daily variance comes overnight

SFM_ch8_range_efficiency

Interpretation: Precision and Bias of the Estimators

	CC	Parkinson	GK	RS	YZ
Efficiency, continuous trading	1.0	5.4	9.3	7.6	8.5
Bias (%), continuous trading	0	−2	−3	−3	−2
Bias (%), overnight jump	1	−22	−22	−22	−2
MSE ratio, overnight jump	1.0	1.8	1.8	1.8	9.0
Bias (%), 26 prices a day	0	−23	−32	−32	−28

- With continuous trading, range estimators are 5 to 9 times more efficient: one month of ranges carries as much information as 5 to 9 months of closes
- With an overnight jump, Parkinson, GK and RS miss the night (bias about −22%); only YZ stays unbiased; **MSE** (mean squared error) ratio = $\mathrm{MSE}(CC)/\mathrm{MSE}$
- With few trades the observed range is too small: all range estimators underestimate (thin markets) Molnár (2012)

SFM_ch8_range_efficiency

Volatility Estimators for Five Assets

	Since	CC	P	GK	RS	YZ	Night share	$\mathrm{corr}(o, c)$
S&P 500	2008	19.9	15.4	14.6	14.4	16.3	13%	0.22
DAX	2006	20.9	16.7	16.6	16.6	20.3	31%	0.02
Bitcoin	2014	66.6	63.5	62.4	62.5	63.2	0%	0.02
TLV	2010	26.6	21.5	20.6	21.0	25.6	26%	−0.08
SNP	2010	24.5	21.4	20.8	21.4	25.8	27%	−0.20

- Annualised volatility (%) on the whole OHLC sample; P = Parkinson; night share $= \hat{\sigma}_O^2/(\hat{\sigma}_O^2 + \hat{\sigma}_C^2)$
- Stocks: Banca Transilvania (TLV) and OMV Petrom (SNP); open, high and low adjusted with the same factor as the close; days without trades removed

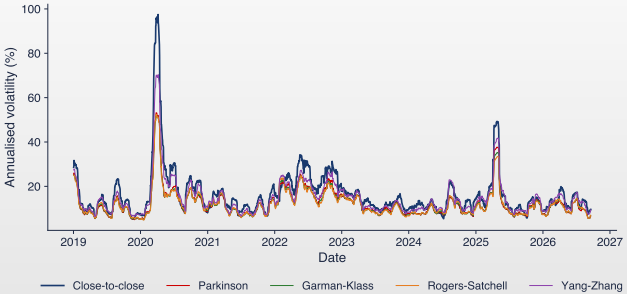
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Interpretation: Differences between the Estimators

- Bitcoin trades 24 hours a day: the night share is almost 0, and all five estimators agree within a few points
  - the remaining gap: the daily path is observed in discrete trades, not continuously
- DAX and the BVB stocks: a quarter to a third of the variance comes overnight
  - Parkinson, GK and RS are far below CC; YZ, which adds the night, is close to CC
- S&P 500: even YZ (16.3%) stays below CC (19.9%)
  - overnight and open-to-close returns are positively correlated (0.22); a likely reason: the index open uses stale prices of stocks that have not traded yet
  - $\mathrm{Var}(r) = \mathrm{Var}(o) + \mathrm{Var}(c) + 2\,\mathrm{Cov}(o, c)$: no range estimator sees the covariance term
- Lesson: check the assumptions on the data before trusting an efficiency gain

SFM_ch8_range_estimators

S&P 500: Rolling 21-Day Estimates since 2019



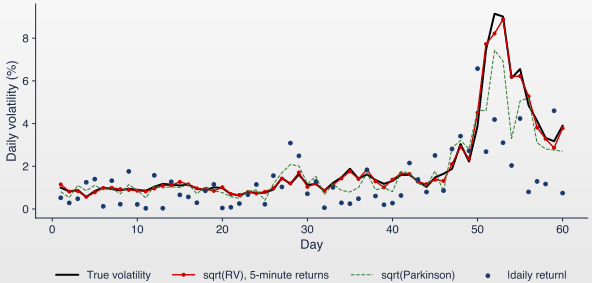
- Annualised 21-day volatility by the five estimators; means since 2019: CC 16.3%, Parkinson 12.5%, YZ 14.7%
- The range-based lines are smoother: the typical daily change of the log estimate is 4.4% for Parkinson and 4.4% for YZ, against 7.0% for CC

SFM_ch8_range_estimators

Realised Variance from Intraday Returns

- Split day t into M intervals (for example $M = 78$ five-minute returns $r_{t,i}$ in a 6.5-hour session)
  - $r_{t,i}$: the log return of interval i of day t, $i = 1, \ldots, M$
  - **realised variance** $\mathrm{RV}_t = \sum_{i=1}^{M} r_{t,i}^2$; **realised volatility** $= \sqrt{\mathrm{RV}_t}$
  - as $M \to \infty$, RV_t converges to the **integrated variance** of day t, the true variance of the day Andersen, Bollerslev, Diebold & Labys (2003), Barndorff-Nielsen & Shephard (2002)
- Volatility becomes almost observable: a nearly error-free measure for each day
  - used to evaluate forecasts (Section 8) and to forecast volatility directly, for example with the HAR model Corsi (2009)
  - high-frequency Bitcoin prices and daily risk: Pele & Mazurencu-Marinescu-Pele (2019)
- Our course data are daily; the next two slides use simulated one-minute prices to show how RV behaves

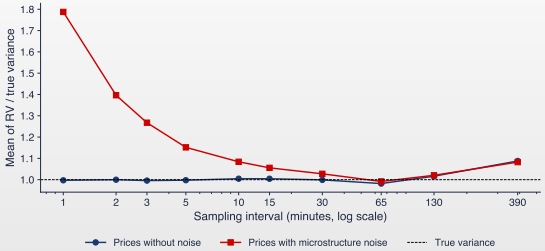
Simulated Example: RV against the True Volatility



- 60 simulated days of 390 one-minute returns; the daily volatility follows a log AR(1) process (persistence 0.9)
- Mean absolute relative error: 5-minute RV 6%, Parkinson 23%, $|r_t|$ 58%; correlation with the truth: 0.995, 0.94 and 0.59

SFM_ch8_realised_volatility

Microstructure Noise and the Signature Plot



- Observed prices = true prices + **microstructure noise** (bid–ask bounce, price ticks), here with sd 0.025%; each squared return then adds 2× the noise variance
- With noise, one-minute RV overstates the variance by a factor of 1.79; at 5 minutes: 1.15; the **signature plot** shows RV against the sampling interval
- In practice, 5-minute RV is hard to beat Liu, Patton & Sheppard (2015)

SFM_ch8_realised_volatility

1982: Engle and the ARCH Model

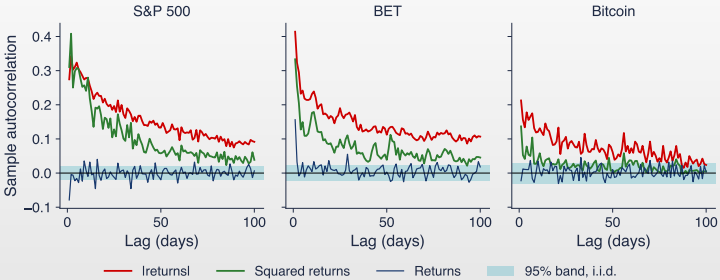
- **Volatility clustering**: periods of large absolute returns alternate with periods of small ones Mandelbrot (1963), Cont (2001)
  - returns are almost uncorrelated (Chapter 7), but $|r_t|$ and r_t^2 are strongly autocorrelated
- Robert F. Engle Engle (1982): the variance of today depends on yesterday's squared shocks: **ARCH**
  - the first model in which σ_t changes over time in a predictable way
  - Nobel Prize in Economic Sciences 2003, shared with Clive Granger Nobel Prize (2003); Engle's Nobel lecture: Engle (2004)
- This section: tests for clustering; Chapter 9: the models



Robert F. Engle (b. 1942), Santiago, 2017

Photo: Econterms (2017); CC BY-SA 4.0; Wikimedia Commons

ACF of Returns, Squared Returns and Absolute Returns



- Sample ACF (autocorrelation function, Chapter 7) of r_t, r_t^2 and $|r_t|$, lags 1–100, with the 95% band $\pm 1.96/\sqrt{T}$ for i.i.d. data
- S&P 500: all 100 autocorrelations of r_t^2 and of $|r_t|$ lie above the band; at lag 100, $|r_t|$ still has 0.09

SFM_ch8_clustering

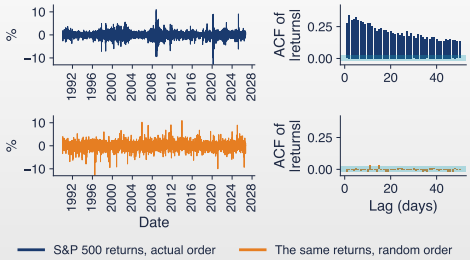
Interpretation: Clustering in Three Markets

	$\hat{\rho}_1(\lvert r\rvert)$	$\hat{\rho}_{10}(\lvert r\rvert)$	$\hat{\rho}_{100}(\lvert r\rvert)$	$\hat{\rho}_1(r^2)$	$\hat{\rho}_{10}(r^2)$	**Lags outside, $\lvert r\rvert$**
S&P 500	0.27	0.28	0.09	0.31	0.24	100/100
BET	0.41	0.23	0.11	0.33	0.18	100/100
Bitcoin	0.21	0.12	0.02	0.14	0.04	92/100

- Slow decay: a shock to volatility is still visible after months; Chapter 11 calls this long memory
- **Taylor effect**: $|r_t|$ is more autocorrelated than r_t^2 (96% of lags for the S&P 500, all lags for the BET and Bitcoin) Ding, Granger & Engle (1993)
- Bitcoin: weaker clustering in r_t^2 (26 of 100 lags outside the band): a few huge days dominate the squares

SFM_ch8_clustering

A Permutation Test for Volatility Clustering



- The same S&P 500 returns in a random order: same distribution and same kurtosis (excess 10.9), but no clustering
- $\hat{\rho}_{\mathbf{1}}(|r|)$: 0.27 in the actual order, -0.02 after shuffling; Ljung–Box $Q(10)$ of $r_t^{\mathbf{2}}$: 8 015 against 6.3
- Clustering is a property of the **order** of the returns, not of their distribution

SFM_ch8_clustering

Testing for Clustering: Ljung–Box on Squared Returns

- H_0: no clustering, $\mathrm{Var}(r_t \mid \mathcal{F}_{t-1})$ constant, so r_t^2 is uncorrelated
  - **McLeod–Li test** McLeod & Li (1983): the Ljung–Box statistic Ljung & Box (1978) applied to r_t^2
  - $Q(m) = T(T+2)\sum_{k=1}^{m} \hat{\rho}_k(r^2)^2/(T-k) \sim \chi^2(m)$ under H_0: a weighted sum of the first m squared autocorrelations
  - T: number of returns; m: number of lags tested; $\hat{\rho}_k(r^2)$: sample autocorrelation of r_t^2 at lag k
  - $Q(m) \geq 0$; large values mean autocorrelated squares; reject if $Q(m)$ exceeds the 95% quantile $\chi^2_{0.95}(m)$
- **Worked example**: S&P 500, $T = 9\,240$, $m = 3$
  - $\hat{\rho}_1, \hat{\rho}_2, \hat{\rho}_3$ of r_t^2: 0.311, 0.408, 0.250
  - terms $T(T+2)\hat{\rho}_k^2/(T-k)$: 892, 1 540, 579; $Q(3) = 3\,012 \gg \chi^2_{0.95}(3) = 7.81$: reject H_0
- The same test on r_t (Chapter 7) asks a different question: is the **mean** predictable?

SFM_ch8_clustering

The ARCH-LM Test (Engle, 1982)

- Auxiliary regression of the squared demeaned returns e_t^2, $e_t = r_t - \bar{r}$, on q of their own lags Engle (1982):
  - $e_t^2 = c_0 + c_1 e_{t-1}^2 + \cdots + c_q e_{t-q}^2 + u_t$; H_0: $c_1 = \cdots = c_q = 0$
  - $\bar{r}$: the sample mean; $c_0, \ldots, c_q$: regression coefficients; u_t: the error term
  - **LM** (Lagrange multiplier) statistic: $\mathrm{LM} = nR^2 \sim \chi^2(q)$, n = number of observations in the regression
- S&P 500, $q = 5$: $R^2 = 0.241$, $n = 9\,235$
  - $R^2 \in [0, 1]$: the share of the variance of e_t^2 explained by its own past
  - $\mathrm{LM} = 9\,235 \times 0.241 = 2\,227 \gg \chi^2_{0.95}(5) = 11.07$: strong ARCH effects
- Interpretation of R^2: about a quarter of the day-to-day variation of e_t^2 is predictable from the last five days
- Both tests are diagnostics: they say that clustering exists, not how to model it (Chapter 9)

SFM_ch8_clustering

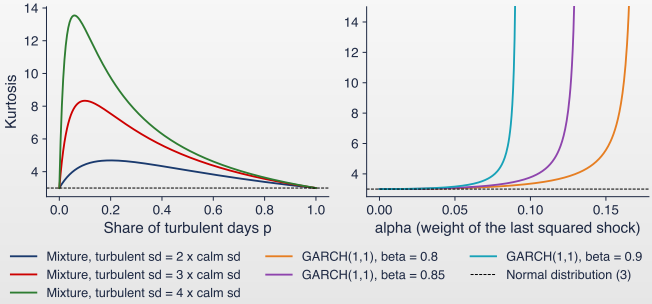
Clustering Tests in Six Series

	T	$\hat{\rho}_1(r)$	$Q(10)$, r	$\hat{\rho}_1(r^2)$	$Q(10)$, r^2	$Q(10)$, $\lvert r\rvert$	ARCH-LM(5)	Exc. kurt.
S&P 500	9 240	−0.079	90.8	0.311	8 015	8 323	2 227	10.9
DAX	9 288	−0.009	21.8	0.155	3 747	5 794	1 209	6.0
BET	7 243	0.156	198.3	0.333	2 500	5 184	1 057	9.9
Bitcoin	4 384	−0.022	20.2	0.137	213	1 086	115	12.0
TLV	3 788	0.010	29.1	0.122	481	1 179	253	13.5
SNP	3 815	−0.015	20.7	0.195	505	944	234	8.9

- 5% critical values: $\chi^2_{0.95}(10) = 18.31$ for $Q(10)$, $\chi^2_{0.95}(5) = 11.07$ for ARCH-LM(5); every statistic on r^2 and $|r|$ rejects with $p < 0.001$
- $Q(10)$ on r also rejects at 5% in all six series ($p \leq 0.028$), but it is far smaller than on r^2 (by a factor of 10.5 to 171.7): mean predictability is weak (Chapter 7), size predictability is strong everywhere
- Bitcoin, Banca Transilvania (TLV) and OMV Petrom (SNP) have the smallest ARCH-LM values despite heavy tails: $\mathrm{LM} = nR^2$ grows with the sample length, and these series are shorter; their R^2 is also lower (0.026, 0.067, 0.061 against 0.241 for the S&P 500): a few extreme days dilute the regression

SFM_ch8_clustering

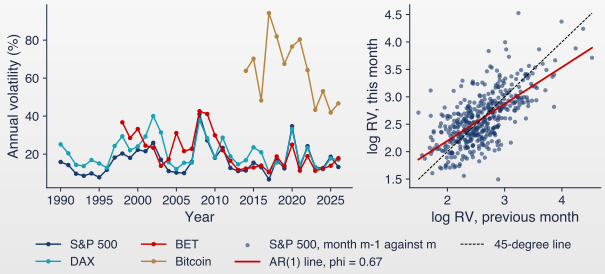
Volatility Clustering and Heavy Tails



- Left: calm days with sd 1, turbulent days with sd 2 (share $p = 0.2$): $E[\sigma^2] = 1.6$, $E[\sigma^4] = 4.0$, kurtosis $3E[\sigma^4]/E[\sigma^2]^2 = 4.69$, though each day is Normal
- Right: GARCH(1,1) kurtosis $3 + 6\alpha^2/(1 - \beta^2 - 2\alpha\beta - 3\alpha^2)$, as in SFEkurgarch; $\alpha = 0.10$, $\beta = 0.85$: 3.77 (Chapter 9)

SFM_ch8_kurgarch

Volatility over Three Decades



- Left: volatility of each calendar year (annualised, %); right: log monthly realised volatility of the S&P 500 (from daily returns) against the previous month
- S&P 500: from 7% (2017) to 41% (2008); BET: 10% (2017) to 43% (2008); Bitcoin: 42% (2025) to 94% (2017)

SFM_ch8_long_term_vix

Interpretation: Persistence and Mean Reversion

- Volatility changes over time, but it does not drift away: it returns to a normal level Schwert (1989)
  - AR(1) of log monthly realised volatility of the S&P 500 (441 months): $\hat{\phi} = 0.67$ (SE 0.04)
  - $\hat{\phi}$: the estimated weight of last month's log volatility; close to 1 means slow mean reversion
  - half-life of a volatility shock: $\ln 0.5 / \ln \hat{\phi} = 1.7$ months; long-run level about 13.5%
- Monthly volatility is right-skewed (skewness 3.3), its logarithm much less (0.6) Andersen, Bollerslev, Diebold & Ebens (2001)
  - mean 15.3% above the median 12.9%: a few crisis months pull the mean up
  - models of volatility are therefore often written for $\ln \sigma_t$
- Mean reversion is what EWMA lacks and GARCH with $\alpha + \beta < 1$ has (Chapter 9)

SFM_ch8_long_term_vix

The VIX: Implied Volatility

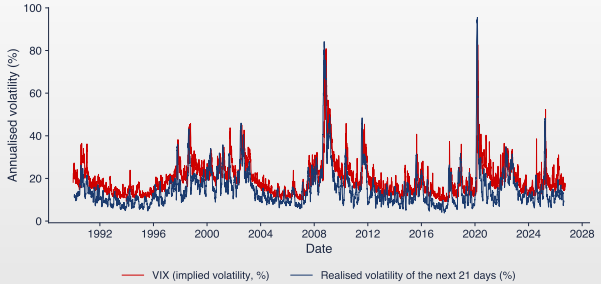
- Options trade on exchanges since 1973, when members of the Chicago Board of Trade founded the **CBOE** (Chicago Board Options Exchange)
  - an option price depends on the volatility the market expects until expiry
  - **implied volatility**: the volatility that, put into an option-pricing formula, gives the observed price
- **VIX**, introduced in 1993 Whaley (1993): the expected volatility of the S&P 500 over the next 30 calendar days, from S&P 500 option prices Cboe (2026)
  - quoted in annualised %: VIX $= 20$ means about $20/\sqrt{12} \approx 5.8\%$ over a month
  - called the "fear index": it jumps when stocks fall Whaley (2009)



Traders at the Chicago Board of Trade, 31 May 1973

Photo: U.S. Department of Agriculture (1973); public domain; Wikimedia Commons

VIX against the Volatility That Followed



- VIX and the realised volatility of the S&P 500 over the next 21 trading days, $\sqrt{(A/21)\sum_{j=1}^{21} r_{t+j}^2}$; maximum VIX: 82.7 on 16 March 2020
- **Question for the room**: is the red line usually above or below the blue one?

SFM_ch8_long_term_vix

Interpretation: Implied against Realised Volatility

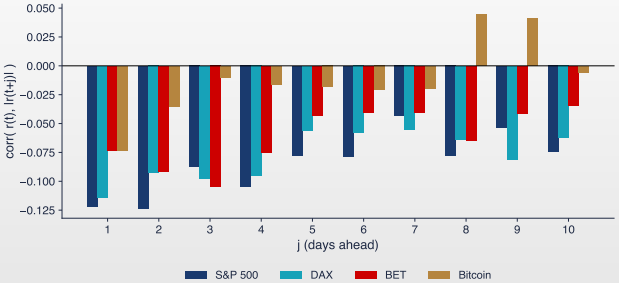
- **Answer**: above; VIX exceeded the next 21 days' realised volatility on 86% of the 9 199 days
  - mean VIX 19.4% against mean realised 15.3%: a gap of 4.1 points
  - **variance risk premium**: investors pay to be insured against volatility spikes Carr & Wu (2009), Bollerslev, Tauchen & Zhou (2009)
- As a forecast, VIX is informative but biased
  - regression of future on implied: realised $= -1.72 + 0.88\,\text{VIX}$, $R^2 = 0.52$
  - the past 21-day realised volatility explains less: $R^2 = 0.44$
- Daily log changes of the VIX and S&P 500 returns: correlation -0.71: fear rises when prices fall

SFM_ch8_long_term_vix

The Leverage Effect

- **Leverage effect**: negative returns are followed by higher volatility than positive returns of the same size Christie (1982)
  - leverage story: a falling share price raises the debt-to-equity ratio, so the equity becomes riskier
  - volatility-feedback story: higher expected volatility raises the required return, so the price falls today French, Schwert & Stambaugh (1987), Bekaert & Wu (2000)
- A simple measure: $\mathrm{corr}(r_t, |r_{t+j}|)$ for $j = 1, 2, \ldots$
  - j: the number of days ahead; corr: the sample correlation over all days t
  - negative: today's fall announces tomorrow's turbulence; zero under a symmetric model
- Chapter 2 met the asymmetry of the distribution; here the asymmetry is in the **dynamics** of volatility

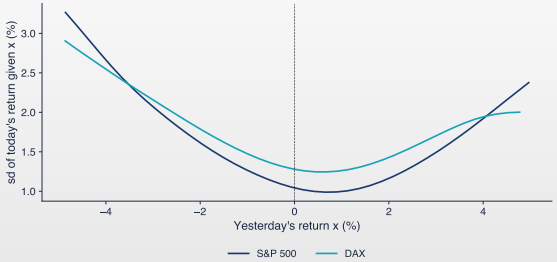
Negative Returns and Future Volatility



- $\mathrm{corr}(r_t, |r_{t+j}|)$, $j = 1, \ldots, 10$; S&P 500: -0.122 at $j = 1$; the reverse direction $\mathrm{corr}(|r_{t-1}|, r_t) = 0.025$
- Mean r^2_{t+1} after a down day divided by that after an up day: S&P 500 1.51, DAX 1.46, BET 1.41, Bitcoin 1.06

SFM_ch8_leverage

Nonparametric Volatility Given Yesterday's Return



- $\hat{\sigma}(x) = \sqrt{\hat{m}_{\mathbf{2}}(x) - \hat{m}_{\mathbf{1}}(x)^{\mathbf{2}}}$, $\hat{m}_{\mathbf{1}}$, $\hat{m}_{\mathbf{2}}$: local linear regressions of r_t and $r_t^{\mathbf{2}}$ on $r_{t-\mathbf{1}} = x$ (quartic kernel, bandwidth 0.04 on decimal returns, as in SFEvolnonparest)
- S&P 500: $\hat{\sigma}(-2\%) = 1.61\%$ against $\hat{\sigma}(+2\%) = 1.16\%$; the minimum is at $x \approx 0.7\%$, not at 0: an asymmetric news impact

SFM_ch8_leverage

The Proxy Problem

- A forecast h_t of σ_t^2 is made at the close of day $t-1$; the true σ_t^2 is never observed
  - we compare h_t with a **proxy** $\hat{\sigma}_t^2$: a measurable quantity with $E[\hat{\sigma}_t^2 \mid \mathcal{F}_{t-1}] = \sigma_t^2$
- Proxies, from noisy to precise: r_t^2, a range estimator, realised variance
  - with r_t^2 even a perfect forecast looks poor: Andersen & Bollerslev (1998) showed that a low R^2 does not mean a bad model
- **Mincer–Zarnowitz regression** Mincer & Zarnowitz (1969): $\hat{\sigma}_t^2 = a + b\, h_t + u_t$
  - a: intercept; b: slope; u_t: error; a good forecast has $a \approx 0$, $b \approx 1$
  - the R^2 measures how much of the proxy the forecast explains

Loss Functions: MSE and QLIKE

- Average loss over the evaluation days; lower is better Patton (2011)
  - L: the loss of one day; h_t: the variance forecast; $\hat{\sigma}_t^2$: the proxy
  - MSE: $L = (\hat{\sigma}_t^2 - h_t)^2$
  - **QLIKE** (quasi-likelihood): $L = \hat{\sigma}_t^2 / h_t + \ln h_t$, minus the Normal log-likelihood up to constants
- Both are **robust**: with a noisy but unbiased proxy they rank forecasts as the true variance would
  - losses on $\sqrt{h_t}$ or on $\ln h_t$ are not robust and can prefer the wrong forecast
- **Worked example**: true variance and proxy equal to 1; two forecasts, $h = 0.5$ and $h = 2$
  - MSE: $(1 - 0.5)^2 = 0.25$ against $(1 - 2)^2 = 1$: MSE punishes over-prediction more
  - QLIKE: $1/0.5 + (-0.693) = 1.307$ against $1/2 + 0.693 = 1.193$: QLIKE punishes under-prediction more, the costly error for risk management

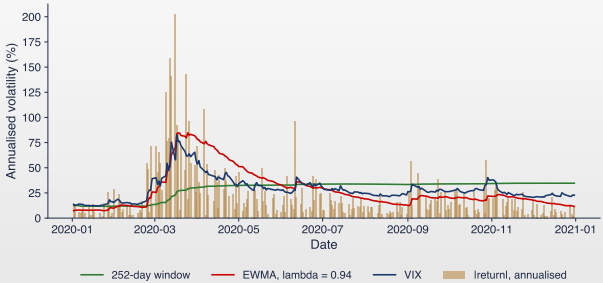
S&P 500: Comparison of the Forecasts

	21 days	63 days	252 days	EWMA 0.94	EWMA 0.97	VIX
MSE	18.99	20.77	21.43	17.77	19.02	16.70
QLIKE	0.876	0.914	1.099	0.808	0.844	0.799
MZ R^2, proxy r^2	0.14	0.05	0.01	0.17	0.11	0.25
MZ R^2, range proxy	0.24	0.09	0.02	0.29	0.19	0.42

- Next-day variance forecasts for 4 202 days from 2010; proxy r_t^2; VIX forecast: VIX^2/A; range proxy: o_t^2 plus the Garman–Klass term
- **DM** (Diebold–Mariano) test of equal QLIKE Diebold & Mariano (1995): EWMA 0.94 against 21 days $t = -4.50$ ($p < 0.001$); VIX against EWMA 0.94 $t = -0.32$ ($p = 0.751$)
- EWMA beats the 21-day window; VIX and EWMA are not significantly different; the long window is worst; a precise proxy raises every R^2, but not the ranking

SFM_ch8_forecast_evaluation

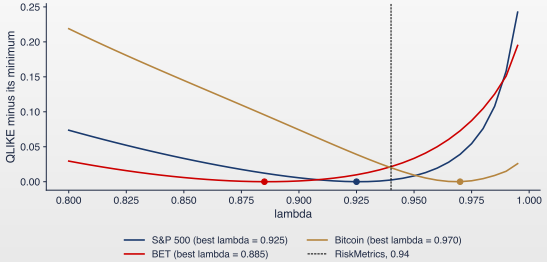
Forecasts in 2020



- Next-day volatility forecasts of the S&P 500 in 2020 (annualised): 252-day window, EWMA ($\lambda = 0.94$) and VIX; bars: $|r_t|$
- VIX started to rise one trading day before the first large fall of February 2020; EWMA rose right after it; the 252-day window moved slowly and stayed near its peak for the rest of the year

SFM_ch8_forecast_evaluation

Choosing λ by QLIKE



- Mean QLIKE of next-day EWMA forecasts (proxy r_t^2, from 2010; Bitcoin from 2016) minus its minimum, for λ from 0.80 to 0.995
- Best λ: S&P 500 0.925, BET 0.885, Bitcoin 0.970; the RiskMetrics 0.94 loses only 0.002 for the S&P 500
- One λ for all assets is a convenient compromise, not an optimum

SFM_ch8_forecast_evaluation

An Open Research Question

- **Do range-based estimators help on the Bucharest Stock Exchange, where many stocks trade thinly?**
  - theory: ranges are 5 to 9 times more efficient; but thin trading shrinks the observed range (Section 4)
  - days with trades but an unchanged close, since 2010: TLV 9%, SNP 8%, BRD 10%, TGN 10%
  - Parkinson / close-to-close volatility: TLV 0.81, SNP 0.87, BRD 0.90, TGN 0.87 (S&P 500: 0.78)
  - QLIKE of next-day forecasts by the 21-day window, CC against YZ: TLV 2.022 against 2.039; TGN 1.750 against 1.636
- Open: the answer differs by stock; is it liquidity, the opening auction, or the size of the overnight move?
- AI tools can speed up such a study, but its results must still be checked Wang et al. (2023)

SFM_ch8_bvb_range

Possible Uses of AI Tools

- **Literature**: list studies of range-based and realised volatility in emerging and thin markets, with their data and estimators
- **Code**: a first draft of the five estimators, rolling forecasts and QLIKE comparisons for all BET stocks
- **Robustness**: liquidity groups by volume, other windows, the 2016–2020 and 2021–2026 halves
- Example prompt
  - `Write a Python function that computes the Parkinson, Garman-Klass, Rogers-Satchell and Yang-Zhang variance on rolling 21-day windows from daily open, high, low and close prices, and compares their next-day forecasts with QLIKE against squared close-to-close returns.`

What to Check

- The constants: $4 \ln 2$ in Parkinson, $2 \ln 2 - 1$ in Garman–Klass, k in Yang–Zhang; a draft with $\ln 2$ or $k = 0.34$ is wrong
- Stock prices: open, high and low must be adjusted for dividends and splits with the same factor as the close
- Annualisation with the actual frequency: 365 for crypto, about 250 for BVB stocks
- No look-ahead: the forecast for day t uses only data up to day $t - 1$
- References: every cited paper must exist; check the DOI

Project Idea

- **Question**: which daily volatility estimator should a Romanian risk manager use?
  - data: BVB stocks since 2010 (TLV, SNP, BRD, TGN, and others with OHLC data), S&P 500 and DAX as benchmarks (EODHD)
- Steps
  - the five estimators on 21-day windows; EWMA versions of each; next-day forecasts
  - QLIKE and MSE with the proxy r_t^2; Diebold–Mariano tests; results by liquidity group
  - relate the gain of YZ over CC to the share of zero-volume days and to the overnight share
- Deliverables: one table, one chart, and a paragraph on what the data can and cannot show
- Declare any AI use, and list the errors of the AI that you corrected

Key Takeaways

- Volatility is the standard deviation of returns; it is latent and changes over time; annualise with the actual frequency
- Historical windows trade speed against noise; EWMA ($\lambda = 0.94$) reacts fast and avoids ghosts
- Range estimators use the daily path: far more efficient, but biased by overnight jumps and thin trading; Yang–Zhang adds the night
- Clustering: $|r_t|$ and r_t^2 are autocorrelated; detect it with Ljung–Box on r_t^2 and ARCH-LM
- Volatility mean-reverts; VIX usually exceeds realised volatility; falls raise future volatility
- Compare forecasts with robust losses (QLIKE) and a proxy; test the differences

Key Formulas

Quantity	**Formula**
Annualised volatility	$\sigma_{\text{year}} = \sigma_{\text{day}}\sqrt{A}$
EWMA	$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)r_{t-1}^2$, half-life $\ln 0.5/\ln\lambda$
Parkinson	$(u_t - d_t)^2/(4\ln 2)$
Garman–Klass	$0.5(u_t - d_t)^2 - (2\ln 2 - 1)c_t^2$
Rogers–Satchell	$u_t(u_t - c_t) + d_t(d_t - c_t)$
Yang–Zhang	$\hat\sigma_O^2 + k\hat\sigma_C^2 + (1-k)\hat\sigma_{RS}^2$, $k = 0.34/(1.34 + (n+1)/(n-1))$
Realised variance	$\mathrm{RV}_t = \sum_i r_{t,i}^2$
McLeod–Li	$Q(m) = T(T+2)\sum_{k=1}^{m}\hat\rho_k(r^2)^2/(T-k) \sim \chi^2(m)$
ARCH-LM	$\mathrm{LM} = nR^2 \sim \chi^2(q)$
MSE, QLIKE	$(\hat\sigma_t^2 - h_t)^2$, $\hat\sigma_t^2/h_t + \ln h_t$

Self-Assessment

- **Question**: a stock has $H = 102$, $L = 98$ on one day; what is the Parkinson daily volatility?
  - **Answer**: $\ln(102/98) = 4.00\%$; $\sqrt{4.00^2/2.77} = 2.40\%$
- **Question**: Ljung–Box on returns does not reject, on squared returns it rejects strongly; what do you conclude?
  - **Answer**: the mean is unpredictable, the variance is predictable: volatility clustering without autocorrelation
- **Question**: why can EWMA with $\lambda = 0.94$ not forecast a return of volatility to its normal level?
  - **Answer**: $\alpha + \beta = 1$ and $\omega = 0$: its forecasts are flat at every horizon
- Next: Chapter 9, ARCH and GARCH models

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