








Time Series Analysis

Chapter 15: Review and exam preparation



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Acronyms used in this material (1/3)

- **ACF**: Autocorrelation Function
- **ADF**: Augmented Dickey–Fuller (test)
- **AI**: Artificial Intelligence
- **AIC**: Akaike Information Criterion
- **AR**: AutoRegressive (model)
- **ARCH**: AutoRegressive Conditional Heteroskedasticity
- **ARCH-LM**: ARCH Lagrange Multiplier (test)
- **ARFIMA**: AutoRegressive Fractionally Integrated Moving Average (model)
- **ARIMA**: AutoRegressive Integrated Moving Average (model)
- **ARMA**: AutoRegressive Moving Average (model)
- **ASE**: Academia de Studii Economice din București (the Bucharest University of Economic Studies)
- **BEKK**: Baba–Engle–Kraft–Kroner (multivariate GARCH model)
- **BET**: Bucharest Exchange Trading (index)
- **BIC**: Bayesian Information Criterion
- **BNR**: Banca Națională a României (the National Bank of Romania)
- **BVB**: Bursa de Valori București (the Bucharest Stock Exchange)
- **CCC**: Constant Conditional Correlation (model)
- **DCC**: Dynamic Conditional Correlation
- **DHR**: Dynamic Harmonic Regression
- **DLOG**: Difference of the LOGarithm (EViews function)

Acronyms used in this material (2/3)

- **DM**: Diebold–Mariano (test)
- **EC1**: Error Correction term 1
- **EC2**: Error Correction term 2
- **ECM**: Error Correction Model
- **ENTSO-E**: European Network of Transmission System Operators for Electricity
- **EODHD**: EOD Historical Data (market-data provider)
- **FEVD**: Forecast Error Variance Decomposition
- **FRED**: Federal Reserve Economic Data
- **GARCH**: Generalised AutoRegressive Conditional Heteroskedasticity
- **GDP**: Gross Domestic Product
- **GJR**: Glosten–Jagannathan–Runkle (GARCH model)
- **GPH**: Geweke–Porter-Hudak (estimator)
- **HEGY**: Hylleberg–Engle–Granger–Yoo (seasonal unit-root test)
- **HICP**: Harmonised Index of Consumer Prices
- **HLN**: Harvey–Leybourne–Newbold (small-sample correction of the DM test)
- **HP**: Hodrick–Prescott (filter)
- **IPC**: Indicele prețurilor de consum — the consumer price index (CPI)
- **IRF**: Impulse Response Function
- **KPSS**: Kwiatkowski–Phillips–Schmidt–Shin (test)
- **LB**: Ljung–Box (test)
- **LPPL**: Log-Periodic Power Law

Acronyms used in this material (3/3)

- **LPPLS**: Log-Periodic Power Law Singularity (model)
- **LSTM**: Long Short-Term Memory (network)
- **MA**: Moving Average
- **MAE**: Mean Absolute Error
- **MAPE**: Mean Absolute Percentage Error
- **MASE**: Mean Absolute Scaled Error
- **NBER**: National Bureau of Economic Research
- **OCSB**: Osborn–Chui–Smith–Birchenhall (seasonal unit-root test)
- **OLS**: Ordinary Least Squares
- **PACF**: Partial AutoCorrelation Function
- **PPP**: Purchasing Power Parity
- **QQ**: Quantile–Quantile (plot)
- **RMSE**: Root Mean Squared Error
- **ROBOR**: Romanian Interbank Offer Rate
- **SAR**: Seasonal AutoRegressive term (EViews)
- **SARIMA**: Seasonal ARIMA (model)
- **SE**: Standard Error
- **SES**: Simple Exponential Smoothing
- **SIGMASQ**: SIGMA SQuared, the innovation variance (EViews)
- **US**: United States
- **VAR**: Vector AutoRegression
- **VAT**: Value Added Tax
- **VEC**: Vector (half-vectorised) GARCH model
- **VECM**: Vector Error Correction Model

Today's question and route

- **Question**: what does the whole course say about a time series, and how is it examined?
  - one chapter that ties Chapters 0–14 together
- **Route**
  - Part I: the course map, two recap slides per chapter, a summary of tests and models
  - Part II: the Box–Jenkins method from start to finish on Romanian inflation
  - Part III: the exam (format, grading, eight solved problems), the team project, attendance and quizzes
- Chapters 11–14 are for self-study



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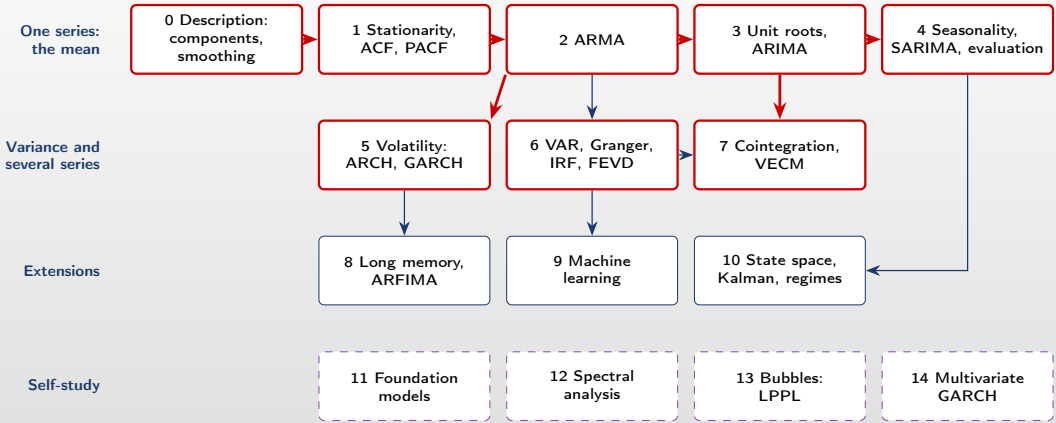
Learning outcomes

- Place every chapter on the path from describing a series to modelling and forecasting it
- Recall the key formula, the key empirical fact and the most common mistake of each chapter
- Choose the right test or model for a given question about a time series
- Run the Box–Jenkins method on a real series: transform, test, identify, estimate, check, forecast
- Read software output (estimates, tests, correlograms) and interpret it in three to five sentences
- Plan the team project, its oral defence and the declaration of AI use

Reading and tools

- Textbook: Huang and Petukhina (2022); free companion: Hyndman and Athanasopoulos (2021)
  - theory: Brockwell and Davis (2016); Hamilton (1994); the method: Box, Jenkins and Reinsel (2008)
- Python Quantlets of this chapter: Quantlets/Ch_15
  - the Box–Jenkins case on the Romanian HICP and the output of exam problem 3
- Lecture notebook, the course in one notebook: open in Google Colab
- Self-assessment: the quiz of this chapter, 20 questions drawn from all chapters

The course map: Chapters 0–14



- Red: the core of the exam, from describing one series to modelling several series together
- Blue: extensions (Chapters 8–10); dashed: self-study (Chapters 11–14), each built on a core chapter: 11 on 9, 12 on 1 and 8, 13 on 3, 14 on 5 and 6

The series of the course

Series	Source	Topic illustrated	Ch.
Romanian real GDP (quarterly)	Eurostat	trend, seasonality, a unit root, regimes	0–4, 10
Romanian HICP inflation (monthly)	Eurostat	persistence, seasonality, long memory	2–4, 8
ROBOR 3M, unemployment	Eurostat	a VAR for the Romanian economy	6
EUR/RON	BNR	a random walk that is hard to beat	0, 3, 5
BET, S&P 500, DAX (daily)	EODHD	volatility clustering, spillovers	1, 5, 6
electricity load (daily, hourly)	ENTSO-E	multiple seasonality, machine learning	4, 9
Nile, sunspots, US yields and GDP	statsmodels, FRED	the classic examples of the textbooks	1, 2, 7, 8, 10

Chapter 0: Introduction: components and exponential smoothing (1/2)

- **Key formulas**
  - additive $y_t = T_t + S_t + R_t$ or multiplicative $y_t = T_t S_t R_t$; the logarithm turns one into the other
  - SES $\ell_t = \alpha y_t + (1 - \alpha)\ell_{t-1}$; MASE = MAE / MAE of the in-sample seasonal naive method
- **Notation**
  - T_t, S_t, R_t: the trend, the seasonal component and the remainder (the irregular part)
  - ℓ_t: the smoothed level at time t; $\alpha \in (0, 1)$: the weight of the newest observation; a large α follows the data closely
  - MAE: the mean absolute error; MASE below 1: better than the seasonal naive method

Chapter 0: Introduction: components and exponential smoothing (2/2)

- **Empirical facts**
  - Romanian GDP grew 2.3 times since 1995 (2.8% a year); Q1 is about -23.8% and Q4 $+20.7\%$ around the trend
  - EUR/RON: $\hat{\alpha}_{SES} = 1.00$, the last value is the best forecast; electricity: Holt–Winters MASE 0.51 against 0.82 for the naive method
- **Common mistakes**
  - judging a forecast on the data used to fit it
  - MAPE for series that cross zero (monthly inflation); an additive model for a seasonal swing that grows with the level

Chapter 1: Stochastic processes and stationarity (1/2)

- **Key formulas**
  - weak stationarity: $EX_t = \mu$, $\mathrm{Cov}(X_t, X_{t+h}) = \gamma(h)$; $\hat{\rho}(h)$ against $\pm 1.96/\sqrt{T}$
  - Ljung–Box $Q^*(m) = T(T+2)\sum_{h=1}^{m} \hat{\rho}(h)^2/(T-h) \sim \chi^2(m)$ Ljung and Box (1978); random walk $\mathrm{Var}(X_t) = t\sigma^2$
- **Notation**
  - μ: the constant mean; $\gamma(h)$: the autocovariance at lag h, which depends only on h; $\hat{\rho}(h)$: the sample autocorrelation; T: the number of observations
  - $Q^*(m)$: the Ljung–Box statistic on the first m lags; a large value (a small p-value) rejects white noise
  - σ^2: the variance of the random-walk shocks; $\mathrm{Var}(X_t) = t\sigma^2$ grows with t, so a random walk is not stationary

Chapter 1: Stochastic processes and stationarity (2/2)

- **Empirical facts**
  - BET returns: $\hat{\rho}_1 = 0.11$, absolute returns $\hat{\rho}_1 = 0.40$ (band ± 0.024): uncorrelated is not independent
  - Romanian GDP: Box–Cox $\hat{\lambda} = 0.02$, the logarithm; one difference too many gives $\hat{\rho}_1 = -0.54$
- **Common mistakes**
  - reading one bar outside the band as a signal (one in twenty is outside by chance)
  - differencing a series that is already stationary

Chapter 2: ARMA models (1/2)

- **Key formulas**
  - $\phi(L)(X_t - \mu) = \theta(L)\varepsilon_t$; AR(1) $\rho(h) = \phi^h$; MA(1) $\rho(1) = \theta/(1+\theta^2)$
  - AIC $= -2\ln\hat{\mathcal{L}} + 2k$ Akaike (1974), BIC $= -2\ln\hat{\mathcal{L}} + k\ln T$ Schwarz (1978); residual check $Q^*(m) \sim \chi^2(m-p-q)$
- **Notation**
  - L: the lag operator, $LX_t = X_{t-1}$; $\phi(L)$, $\theta(L)$: the AR and MA polynomials of orders p and q; ε_t: white noise
  - $\ln\hat{\mathcal{L}}$: the maximised log-likelihood; k: the number of estimated parameters; a smaller AIC or BIC is better
  - BIC penalises parameters more than AIC, since $\ln T > 2$ for $T \geq 8$

Chapter 2: ARMA models (2/2)

- **Empirical facts**
  - annual growth of Romanian GDP: MA(3) by BIC, a product of overlapping quarters; residual Ljung–Box $p = 0.37$
  - Romanian inflation: AR(2) with $\hat{\phi}_1 + \hat{\phi}_2 = 0.976$, close to a unit root; sunspots: an AR(2) cycle of 10.6 years
- **Common mistakes**
  - reading the AR order from the ACF (it is the PACF that cuts off for an AR)
  - Ljung–Box on residuals with m instead of $m - p - q$ degrees of freedom; a common factor in $\phi(L)$ and $\theta(L)$

Chapter 3: Unit roots and ARIMA models (1/2)

- **Key formulas**
  - ADF: $\Delta y_t = c + bt + \gamma y_{t-\mathbf{1}} + \sum_j \delta_j \Delta y_{t-j} + \varepsilon_t$, $H_{\mathbf{0}}$: $\gamma = 0$ Dickey and Fuller (1979); KPSS: $H_{\mathbf{0}}$ stationarity Kwiatkowski et al. (1992)
  - ARIMA(p, d, q): $\phi(L)(1 - L)^d y_t = c + \theta(L)\varepsilon_t$; forecast variance grows with h when $d \geq 1$
- **Notation**
  - $\Delta y_t = y_t - y_{t-\mathbf{1}}$; c: the constant; bt: the deterministic trend; $\gamma = 0$ means a unit root
  - δ_j: the coefficients of the lagged differences, which remove the autocorrelation of the errors
  - d: the number of differences needed for stationarity; $(1 - L)^d$: the difference of order d; h: the forecast horizon

Chapter 3: Unit roots and ARIMA models (2/2)

- **Empirical facts**
  - log Romanian GDP: $\tau = -2.31$ ($p = 0.43$, 5% critical value -3.45); growth: $\tau = -10.77$; ARIMA(0,1,0) with drift 0.81% a quarter
  - two independent random walks: a significant t in 90% of regressions Granger and Newbold (1974); ADF power only 12.5% for $\phi = 0.95$, $T = 100$
- **Common mistakes**
  - Student or Normal critical values for the ADF statistic
  - reading a non-rejection as proof of a unit root; a regression in levels of $I(1)$ series without a cointegration test

Chapter 4: Seasonality and forecasting (1/2)

- **Key formulas**
  - SARIMA $\phi(L)\Phi(L^s)(1-L)^d(1-L^s)^D y_t = \theta(L)\Theta(L^s)\varepsilon_t$; airline $(0,1,1)(0,1,1)_s$
  - Diebold–Mariano $\bar{d}/\sqrt{\hat{V}/n}$, $d_t = L(e_{1t}) - L(e_{2t})$ Diebold and Mariano (1995), with the HLN correction Harvey, Leybourne and Newbold (1997)
- **Notation**
  - s: the season length (12 for monthly data); $\Phi(L^s)$, $\Theta(L^s)$: the seasonal AR and MA polynomials; D: the number of seasonal differences $(1-L^s)$
  - e_{1t}, e_{2t}: the errors of the two forecasts; $L(\cdot)$: the loss, e.g. the square; d_t: the loss differential; $\bar{d}$: its mean; $\hat{V}$: its estimated long-run variance; n: the number of forecasts
  - under equal accuracy DM is approximately $N(0,1)$: $|\text{DM}| > 1.96$ rejects at 5%

Chapter 4: Seasonality and forecasting (2/2)

- **Empirical facts**
  - airline passengers: MAPE 8.5% for the airline model against 15.5% for the seasonal naive method
  - Orthodox Easter raises Romanian food sales by about 5.9%; daily load: DHR MASE 0.89 against 1.28 (seasonal naive)
- **Common mistakes**
  - comparing AICc of models with different d or D
  - judging forecasts at a single origin; seasonally adjusted data in a SARIMA forecast of raw data

Chapter 5: Conditional volatility: ARCH and GARCH (1/2)

- **Key formulas**
  - GARCH(1,1) $\sigma_t^2 = \omega + \alpha\varepsilon_{t-1}^2 + \beta\sigma_{t-1}^2$ Engle (1982), Bollerslev (1986); $\bar{\sigma}^2 = \omega/(1-\alpha-\beta)$, $h_{1/2} = \ln 0.5/\ln(\alpha+\beta)$
  - ARCH-LM $= nR^2 \sim \chi^2(q)$; VaR 1% $= -(\mu + \sigma_{t+1} q_{0.01}(z))$
- **Notation**
  - σ_t^2: the conditional variance; ε_{t-1}: yesterday's shock; $\omega > 0$: the constant; $\alpha \geq 0$: the reaction to shocks; $\beta \geq 0$: the persistence
  - $\bar{\sigma}^2$: the long-run variance, defined only if $\alpha + \beta < 1$; $h_{1/2}$: the half-life of a volatility shock
  - ARCH-LM: R^2 of the regression of ε_t^2 on q of its lags, n observations; $q_{0.01}(z)$: the 1% quantile of the standardised innovations z_t; μ: the conditional mean

Chapter 5: Conditional volatility: ARCH and GARCH (2/2)

- **Empirical facts**
  - GARCH(1,1)-t: $\alpha + \beta = 0.995$ for the S&P 500 (half-life 131 days), 0.991 for the BET (74 days)
  - S&P 500: GJR $\hat{\gamma} = 0.21$ ($t = 10.0$); kurtosis 13.7 for returns, 5.0 for standardised residuals
- **Common mistakes**
  - the half-life written as $1/(1 - \alpha - \beta)$; a long-run variance reported when $\alpha + \beta \geq 1$
  - "VaR 99%" or a negative VaR: the course writes VaR 1%, a positive loss

Chapter 6: VAR models and Granger causality (1/2)

- **Key formulas**
  - VAR(p): $\mathbf{Y}_t = \mathbf{c} + \mathbf{A_1}\mathbf{Y}_{t-\mathbf{1}} + \cdots + \mathbf{A}_p\mathbf{Y}_{t-p} + \boldsymbol{\varepsilon}_t$ Sims (1980); stable if all eigenvalues of the companion matrix are inside the unit circle
  - Granger $F = [(RSS_R - RSS_U)/p] \,/\, [RSS_U/(T - Kp - 1)]$ Granger (1969); IRF and FEVD with a Cholesky ordering
- **Notation**
  - $\mathbf{Y}_t$: the vector of the K variables; $\mathbf{c}$: the constants; $\mathbf{A}_i$: the $K \times K$ coefficient matrices of lag i; $\boldsymbol{\varepsilon}_t$: the vector of shocks
  - the companion matrix writes the VAR(p) as a VAR(1) of dimension Kp
  - RSS_R, RSS_U: the residual sums of squares without and with the lags of the tested variable; p: the number of restrictions

Chapter 6: VAR models and Granger causality (2/2)

- **Empirical facts**
  - Romanian VAR(2) of growth, inflation and ROBOR 3M: growth Granger-causes ROBOR, $F = 7.26$, $p = 0.001$
  - spillovers among the S&P 500, DAX and BET: 26.4% on average, 57.5% on 17 March 2020
- **Common mistakes**
  - reading Granger causality as causality (it is extra predictability)
  - IRFs reported without the ordering behind them; too many lags for a short sample

Chapter 7: Cointegration and VECM (1/2)

- **Key formulas**
  - Engle–Granger: OLS in levels, then ADF on residuals with MacKinnon critical values Engle and Granger (1987); ECM half-life $\ln 0.5 / \ln(1+\gamma)$
  - VECM $\Delta \mathbf{y}_t = \alpha \beta^\top \mathbf{y}_{t-\mathbf{1}} + \sum_i \Gamma_i \Delta \mathbf{y}_{t-i} + \mathbf{u}_t$; trace $-T \sum_{i>r} \ln(1-\hat{\lambda}_i)$ Johansen (1991)
- **Notation**
  - ECM: $\gamma < 0$ is the speed of adjustment towards equilibrium
  - β: the cointegrating vectors (the equilibrium relations); α: the adjustment coefficients; Γ_i: the short-run dynamics; $\mathbf{u}_t$: the shocks
  - $\hat{\lambda}_i$: the estimated eigenvalues, in decreasing order; r: the tested cointegration rank; a trace above its critical value rejects rank r

Chapter 7: Cointegration and VECM (2/2)

- **Empirical facts**
  - US yields at 1, 5 and 10 years: trace $54.6 > 29.8$ for $r = 0$, $3.12 < 3.84$ for $r \leq 2$: rank 2; long-rate ECM half-life 20.8 months
  - PPP for the leu: ADF $p = 0.24$ on the real exchange rate, not stationary; pairs trading on the BVB: Sharpe 0.15 gross, 0.05 net
- **Common mistakes**
  - Dickey–Fuller critical values for Engle–Granger residuals (they must be more negative)
  - a VAR in differences for cointegrated series (it omits the error correction term)

Chapter 8: Long memory and ARFIMA (1/2)

- **Key formulas**
  - $(1-L)^d$ with $\pi_k = \pi_{k-\mathbf{1}}(k-1-d)/k$ Granger and Joyeux (1980), Hosking (1981); $\rho(k) \sim Ck^{\mathbf{2}d-\mathbf{1}}$; $H = d + 1/2$
  - stationary for $d < 1/2$, mean-reverting for $d < 1$; GPH and local Whittle use the lowest m frequencies
- **Notation**
  - d: the order of fractional integration; π_k: the weight of lag k in the expansion of $(1-L)^d$
  - $C > 0$: a constant; $\rho(k) \sim Ck^{\mathbf{2}d-\mathbf{1}}$: the ACF decays hyperbolically, not exponentially
  - H: the Hurst exponent, above 0.5 for long memory, estimated here by the rescaled-range (R/S) statistic; m: the number of low frequencies used by GPH and local Whittle

Chapter 8: Long memory and ARFIMA (2/2)

- **Empirical facts**
  - Romanian monthly inflation: ARFIMA(0,d,0) with $\hat{d} = 0.31$ (SE 0.05) beats ARMA by AIC and BIC
  - S&P 500: H (R/S) 0.53 for returns, 0.86 for $|r_t|$; the Nile: $\hat{d} = 0.40$, but -0.17 after the break of 1898
- **Common mistakes**
  - a break or a regime change read as long memory
  - one bandwidth m only; long memory in returns confused with long memory in volatility

Chapter 9: Machine learning for time series (1/2)

- **Key formulas**
  - direct forecast $\hat{y}_{T+h} = \hat{f}_h(y_T, \ldots, y_{T-p+1}, \mathbf{z}_{T+h})$; ridge $+\lambda \sum \beta_j^2$, lasso $+\lambda \sum |\beta_j|$
  - bias–variance $E(y_0 - \hat{f})^2 = \text{bias}^2 + \text{Var}(\hat{f}) + \sigma^2$; walk-forward validation only
- **Notation**
  - $\hat{f}_h$: the estimated function for horizon h; p: the number of lags used; $\mathbf{z}_{T+h}$: known regressors, e.g. the calendar
  - $\lambda \geq 0$: the strength of the penalty; β_j: the coefficients; ridge shrinks them, lasso sets some exactly to 0
  - y_0: a new observation; σ^2: the noise variance, which no model can remove

Chapter 9: Machine learning for time series (2/2)

- **Empirical facts**
  - S&P 500, the same model: $R^2 = 0.37$ with random k-fold, -0.39 walk-forward: leakage
  - sign of returns: majority class 54.4%, boosting 52.5%; daily load: ridge MASE 0.90, LSTM 1.16 Makridakis, Spiliotis and Assimakopoulos (2020)
- **Common mistakes**
  - random cross-validation and scaling on the whole sample (look-ahead)
  - trees on trending levels (they cannot extrapolate); no simple benchmark in the comparison

Chapter 10: State space models, Kalman filter and Markov switching (1/2)

- **Key formulas**
  - $y_t = Z\alpha_t + \varepsilon_t$, $\alpha_{t+1} = T\alpha_t + R\eta_t$; $K_t = P_t Z' F_t^{-1}$, $a_{t|t} = a_t + K_t v_t$ Kalman (1960)
  - Markov switching: durations $1/(1-p_{ii})$, ergodic $\pi_1 = (1-p_{22})/(2-p_{11}-p_{22})$ Hamilton (1989)
- **Notation**
  - α_t: the hidden state (e.g. the level); Z, T, R: known system matrices; ε_t, η_t: the measurement and state noise
  - a_t, P_t: the predicted state and its variance; $v_t = y_t - Za_t$: the prediction error, with variance F_t; K_t: the Kalman gain, the weight of the surprise
  - p_{ii}: the probability of staying in regime i; π_1: the long-run share of regime 1

Chapter 10: State space models, Kalman filter and Markov switching (2/2)

- **Empirical facts**
  - the Nile: $\hat{q} = \hat{\sigma}^2_\eta / \hat{\sigma}^2_\varepsilon = 0.097$, steady-state gain $\bar{K} = 0.267$, the weight of exponential smoothing
  - US GDP: the regimes match the NBER dates in 90% of quarters; Romanian GDP: stable regime 15.2 quarters, volatile 11.2
- **Common mistakes**
  - smoothed probabilities used as if known in real time
  - durations written as $p_{ii}/(1 - p_{ii})$; the last value of the HP gap read as a real-time estimate

Chapter 11: Foundation models for time series (self-study)

- **Key ideas**
  - transformers pre-trained on many series: tokenisation and patching of the values
  - Chronos, TimesFM, Moirai, Lag-Llama: zero-shot forecasts and fine-tuning
  - a fair comparison: the same test sample, MASE, probabilistic scores, simple benchmarks
- **Link with the core of the course**
  - a global model of many series (Chapter 9); judged like any forecast (Chapter 4)
- Chapters 11–14 are for self-study

Chapter 12: Spectral analysis (self-study)

- **Key ideas**
  - the Fourier transform, the periodogram $I(\nu_j)$ and the spectral density
  - smoothed periodograms (Welch), filters (Hodrick–Prescott) and business cycles
  - coherence between two series; wavelets as a time–frequency tool
- **Link with the core of the course**
  - the same information as the ACF, by frequency (Chapter 1); the spectral pole of long memory (Chapter 8)
- Chapters 11–14 are for self-study

Chapter 13: Speculative bubbles: LPPL models (self-study)

- **Key ideas**
  - bubbles as super-exponential growth of prices
  - the LPPLS model: power law plus log-periodic oscillations; estimation and filter conditions
  - the LPPLS confidence indicator on historical bubbles and crashes
- **Link with the core of the course**
  - explosive roots, the opposite of unit roots (Chapter 3); volatility before crashes (Chapter 5)
- Chapters 11–14 are for self-study

Chapter 14: Multivariate GARCH models (self-study)

- **Key ideas**
  - time-varying covariances and correlations; the number of parameters grows fast
  - VEC, BEKK, CCC and DCC; estimation of DCC in two steps
  - dynamic correlations, portfolio variance and VaR 1%
- **Link with the core of the course**
  - GARCH for each series (Chapter 5) and the multivariate view of the VAR (Chapter 6)
- Chapters 11–14 are for self-study

Tests and models (1/2): univariate series and the mean

Question	Tool	Decision	Ch.
Is the series stationary?	plot, ACF; ADF and KPSS	ADF rejects, KPSS does not: stationary	1, 3
Trend: deterministic or stochastic?	ADF and KPSS with a trend	detrend, or difference	3
Is there seasonality?	seasonal plot, ACF at s, $2s$; OCSB, HEGY	Δ_s, or dummies and Fourier terms	4
Which ARMA?	ACF, PACF; AIC, BIC	smallest BIC with white-noise residuals	2
Are the residuals adequate?	Ljung–Box on $m - k$ df; Jarque–Bera	$p > 0.05$: no autocorrelation left	2
Which forecast is better?	test sample, rolling origins; MASE, DM	DM $p < 0.05$; otherwise a tie	0, 4

Tests and models (2/2): variance, multivariate series and extensions

Question	Tool	Decision	Ch.
Does the variance change?	Ljung–Box on r_t^2, ARCH-LM	reject: GARCH(1,1)-t, GJR	5
Does x help to forecast y?	VAR, Granger F test	$p < 0.05$: Granger causality	6
What does a shock do?	IRF, FEVD (Cholesky)	bootstrap bands; try other orderings	6
A long-run equilibrium?	Engle–Granger, Johansen	VECM: β, α, half-life	7
Long memory?	GPH, local Whittle, ARFIMA	several m; check breaks	8
Many predictors, non-linear effects?	ridge, lasso, trees, boosting	walk-forward against a benchmark	9
A hidden level or regime?	Kalman filter, Markov switching	filtered for real time, smoothed for history	10

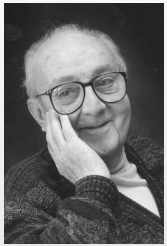
Conventions used throughout the course

- **Transformations**
  - $100 \ln y_t$
  - growth $100\Delta \ln y_t$
  - annual rates $100\Delta_{12} \ln y_t$ (monthly), $100\Delta_4 \ln y_t$ (quarterly)
  - "DLOG(IPC,1,12)" in EViews output is $\Delta\Delta_{12} \ln \text{IPC}$
- **Tests**
  - state H_0, the statistic, its distribution, the critical value or p-value, the decision at 5%
  - then one sentence on what the decision means for the series
- **Model choice**
  - AIC, AICc and BIC only on the same sample and the same d, D
  - forecasts are judged out of sample, against the naive and seasonal naive methods
- **Risk**
  - the level is the tail probability: VaR 1%, a positive loss, $\text{VaR}_\alpha = -q_\alpha$, q_α: the α-quantile of the return

The method in six steps

1. **Plot** and transform: logarithm, Δ, Δ_{12}
2. **Test**: ADF and KPSS for each transformation
3. **Identify**: ACF and PACF of the stationary series
4. **Estimate**: candidate SARIMA models by maximum likelihood; AICc and BIC
5. **Check**: residuals (Ljung–Box, Jarque–Bera, outliers); back to step 3 if they fail
6. **Forecast and evaluate**: a test sample, benchmarks, then intervals

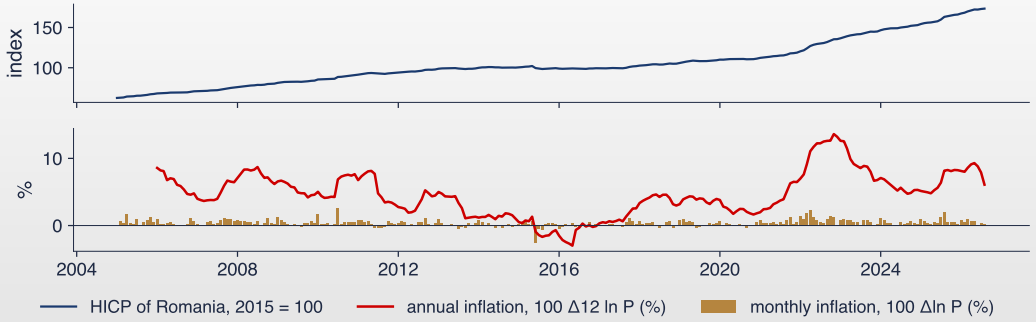
- Data: the HICP of Romania (Eurostat), January 2005 – August 2026, $T = 260$ months; Box, Jenkins and Reinsel (2008)



George E. P. Box (1919–2013)

Photo: DavidMCEddy (2011); CC BY-SA 3.0; Wikimedia Commons

Step 1: the data



⊡ Top: HICP, 2015 = 100; bottom: monthly inflation $100\Delta \ln P_t$ (bars) and annual inflation $100\Delta_{12} \ln P_t$ (line)

TSA_ch15_box_jenkins

Interpreting the data

- **Trend**
  - the index rises almost every month
  - the logarithm is the natural scale
  - monthly inflation averages 0.39%; annual inflation ranges from -3.0% (May 2016) to 13.6% (November 2022)
- **Seasonality**
  - January averages 0.74%, June -0.03% (fresh food)
  - a seasonal pattern in $\Delta \ln P_t$: we will need Δ_{12} or seasonal terms
- **Shocks**
  - tax changes move prices in one month
  - July–August 2025: end of the electricity price caps and a higher VAT; annual inflation from 6.4% to 8.2%

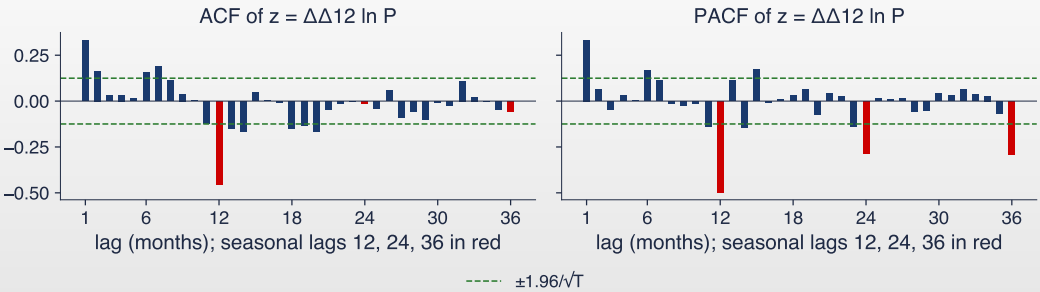
Step 2: unit-root and stationarity tests

Series	Terms	ADF τ	p	KPSS η (5%: crit.)	Verdict
$100 \ln P_t$	const., trend	-0.96	0.949	0.314 (0.146)	unit root
$\Delta \ln P_t$	const.	-4.00	0.001	0.431 (0.463)	stationary, seasonal
$\Delta_{12} \ln P_t$	const.	-1.69	0.435	0.424 (0.463)	inconclusive
$z_t = \Delta\Delta_{12} \ln P_t$	const.	-4.83	< 0.001	0.113 (0.463)	stationary

- ADF: H_0 unit root, lags by AIC Dickey and Fuller (1979); KPSS: H_0 stationarity Kwiatkowski et al. (1992); 5% ADF critical values -3.43 (with trend), -2.87 (constant)
- Annual inflation: neither ADF nor KPSS rejects: an inconclusive result, as in Chapter 3
- Decision: $d = 1$, $D = 1$: we model z_t, both tests agree that it is stationary

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Step 3: identification from the correlogram of z_t



⊡ ACF and PACF of $z_t = \Delta\Delta_{12} \ln P_t$, lags 1–36, $T = 247$; seasonal lags in red; band ± 0.125

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Interpreting the correlogram

- **Regular part**
  - $\hat{\rho}_1 = 0.33$, $\hat{\rho}_2 = 0.16$
  - PACF $\hat{\phi}_{11} = 0.33$, then small: an AR(1)
  - an MA(1) is also possible: ACF and PACF both shrink after lag 1
- **Seasonal part**
  - ACF -0.45 at lag 12, then -0.01 at 24
  - PACF -0.50, -0.28, -0.29 at 12, 24, 36
  - one seasonal ACF spike and a decaying seasonal PACF: a seasonal MA(1)
- First candidate: SARIMA$(1,1,0)(0,1,1)_{12}$ for $\ln P_t$; the satellites at lags 11 and 13 are the product $\rho_1\rho_{12}$

Step 4: estimating the candidates

Model	param.	AICc	BIC	LB(12) p	LB(24) p
$(1,1,0)(0,1,1)_{\mathbf{12}}$, identified	3	295.2	305.3	0.004	0.004
$(1,1,1)(0,1,1)_{\mathbf{12}}$	4	285.4	298.9	0.029	0.088
$(2,1,1)(0,1,1)_{\mathbf{12}}$	5	282.6	299.4	0.233	0.237
$(1,1,2)(0,1,1)_{\mathbf{12}}$	5	282.8	299.6	0.233	0.217
$(1,1,1)(1,1,1)_{\mathbf{12}}$	5	287.7	304.5	0.017	0.070

- All 36 SARIMA$(p,1,q)(P,1,Q)_{\mathbf{12}}$ with $p,q \leq 2$, $P,Q \leq 1$, on the training sample to August 2024 ($T = 236$); first row: the identified model, then the four other models with the lowest BIC, in order
- param.: all estimated parameters, the variance included; Ljung–Box on the residuals with $m - k$ degrees of freedom, $k =$ the ARMA parameters
- The identified model fails the residual check; the lowest BIC, $(1,1,1)(0,1,1)$, fails at lag 12; the chosen model is $(2,1,1)(0,1,1)_{\mathbf{12}}$, the lowest BIC among the models that pass

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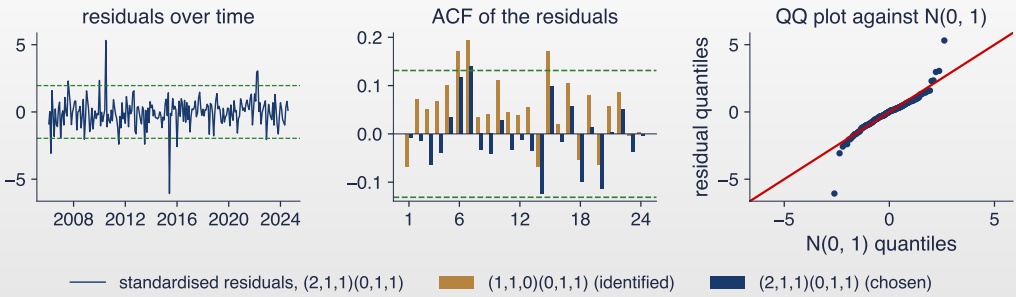
Step 4: the chosen model

Parameter	Estimate	SE	z	p
ϕ_1 (ar.L1)	1.156	0.149	7.76	< 0.001
ϕ_2 (ar.L2)	-0.183	0.122	-1.50	0.133
θ_1 (ma.L1)	-0.856	0.099	-8.61	< 0.001
Θ_1 (ma.S.L12)	-0.935	0.098	-9.55	< 0.001
σ^2	0.179			

- $(1 - \phi_1 L - \phi_2 L^2)\, z_t = (1 + \theta_1 L)(1 + \Theta_1 L^{12})\, \varepsilon_t$, $z_t = \Delta\Delta_{12}\, 100 \ln P_t$
- $\hat{\Theta}_1$ close to -1: the seasonal pattern is almost fixed, Δ_{12} nearly cancels; seasonal dummies would be an alternative
- The smallest AR root has modulus 1.034: inflation shocks are very persistent, as in Chapters 2 and 8

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Step 5: checking the residuals



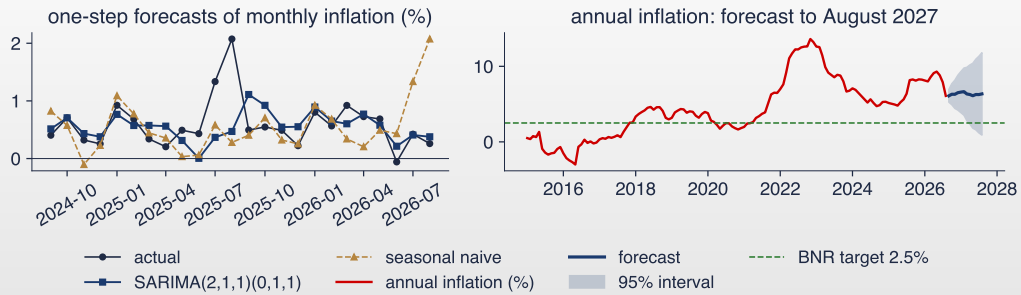
⊡ Standardised residuals of the chosen model; ACF of the residuals of both models; QQ plot against $N(0, 1)$

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Interpreting the residual checks

- **No autocorrelation left**
  - Ljung–Box $Q(12) = 10.49$ (8 df, $p = 0.233$), $Q(24) = 24.13$ ($p = 0.237$)
  - the identified model had $p = 0.004$ and 0.004: the extra AR and MA terms reduce the autocorrelation at lags 6–7
- **Not Normal**
  - Jarque–Bera 450, kurtosis 9.9
  - the two largest residuals: June 2015 (-6.1σ, the VAT cut on food) and July 2010 (5.3σ, the VAT increase)
- Consequence: point forecasts are fine, Normal intervals are too narrow when taxes change; dummies for known tax changes would help

Step 6: forecasts



- Left: one-step forecasts of monthly inflation on the test sample (September 2024 – August 2026), parameters fixed at the training estimates
  - right: annual inflation forecasts to August 2027 with 95% intervals, model re-estimated on all data

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Interpreting the forecasts

Monthly inflation, test sample	RMSE	MAE	MASE
SARIMA$(2,1,1)(0,1,1)_{12}$	0.45	0.28	0.66
seasonal naive	0.64	0.42	0.99
naive	0.51	0.38	0.88

- SARIMA has the smallest errors, but Diebold–Mariano (HLN) does not reject equal accuracy: -1.40 ($p = 0.174$) against the seasonal naive, -0.47 ($p = 0.642$) against the naive method
  - 24 months are few; the largest misses: August 2025 (2.07% against 0.47%) and July 2025 (1.34% against 0.37%): tax and energy decisions that no univariate model can foresee
- Annual inflation: 6.1% in August 2026; forecast 6.1% for September 2026 [5.3, 6.9], 6.3% for August 2027 [0.9, 11.8]
  - the August 2025 VAT jump leaves the 12-month window in August 2026; the 12-month interval is wide: a near unit root in $\Delta \ln P_t$

TSA_ch15_box_jenkins

Recap: Box–Jenkins on Romanian inflation

- Plot, transform, test: $\ln P_t$ needs $d = 1$ and $D = 1$
- The correlogram suggests $(1,1,0)(0,1,1)_{12}$; the residual check rejects it; the chosen model is $(2,1,1)(0,1,1)_{12}$
- Out of sample the gain over the seasonal naive method is not significant; tax shocks dominate the errors

The written exam: format

- **70% of the final grade**; written, 2 hours
  - five subjects, all compulsory; 9 points for the subjects plus 1 point ex officio
- **Three kinds of subjects**
  - a short derivation: moments, stationarity, the ACF or the form of a process
  - the interpretation of software output: estimates, tests, correlograms, VAR and VECM tables
  - model choice and a forecast computed by hand from the output
- The critical values you need are printed in the output, as in the problems below



The National Institute of Statistics, Bucharest: the source of many exam series

Photo: Dan Mihai Pitea (2009); CC BY-SA 3.0; Wikimedia Commons

Content assessed

- **Chapters 0–10**
  - lectures and seminars
  - the definitions and formulas of the "Prerequisites for Today" slides of every seminar
  - the computations of Part A of the seminars, on paper
- **Reading output**
  - tables like those of Part B and of this chapter
  - EViews or Python output: the same numbers under different names (AR(1), ar.L1; SAR(12), ar.S.L12)
  - no code is written at the exam
- **Core weight**
  - ARIMA and SARIMA, unit roots, VAR and Granger causality, cointegration and VECM, GARCH
  - Chapters 11–14 are for self-study

Grading criteria

- **The method**
  - the formula or the equation, with the numbers substituted
  - a correct method with an arithmetic slip receives most of the points
- **The result**
  - the number with its unit (%, percentage points, months) and its sign
  - for a test: H_0, the statistic, the critical value or the p-value, the decision
- **The interpretation**
  - three to five sentences, statistical and economic
  - a correct number with a wrong interpretation does not receive all the points
  - say what the model cannot show (causality, structural breaks, the future of policy)

Typical problem types

Type	Chapters	Example in this chapter
derivation: moments, stationarity, differencing	1–3	Problem 1: trend plus AR(1) errors
correlogram and unit-root output: identification	1–3	Problem 2: Romanian GDP
SARIMA output and a forecast by hand	2–4	Problem 3: Romanian inflation
VAR and Granger output	6	Problem 4: growth, inflation, ROBOR
cointegration and VECM output	7	Problem 5: US yields
GARCH output and VaR 1%	5	Problem 6: the BET
forecast evaluation; state space and regimes	4, 8–10	Problems 7 and 8

- The eight problems below are review examples with full solutions; the exam has its own subjects
- More practice: Part A of every seminar, Seminar 15 and the practice set (25 problems with numerical answers)

Problem 1: a trend with AR(1) errors

Context

- $Y_t = 10 + 0.5t + u_t$, $u_t = 0.6u_{t-\mathbf{1}} + \varepsilon_t$, $\varepsilon_t \sim WN(0,1)$
- At $T = 100$ we observe $Y_T = 62$

Tasks

1. Compute $E(Y_t)$ and $\mathrm{Var}(Y_t)$ and say whether Y_t is stationary.
2. Show that ΔY_t is an ARMA(1,1) with $\theta = -1$ and compute $\rho_{\Delta Y}(1)$.
3. Forecast $Y_{T+\mathbf{1}}$ and $Y_{T+\mathbf{10}}$ with the correct model and give the variance of the 10-step error.
4. Explain what goes wrong if the analyst differences Y_t.

Problem 1: solution

1. $E(Y_t) = 10 + 0.5t$ depends on t: not stationary; $\text{Var}(Y_t) = 1/(1 - 0.36) = 1.5625$ is constant: **trend-stationary**
2. $\Delta Y_t = 0.5 + \Delta u_t$ and $(1 - 0.6L)\Delta u_t = (1 - L)\varepsilon_t$: ARMA(1,1), $\theta = -1$; $\gamma(0) = 1.25$, $\gamma(1) = -0.25$, $\rho(1) = -0.20$
3. $u_T = 62 - 10 - 50 = 2.0$; $\hat{Y}_{T+\mathbf{1}} = 10 + 50.5 + 0.6 \cdot 2 = 61.70$; $\hat{Y}_{T+\mathbf{10}} = 10 + 55 + 0.6^{\mathbf{10}} \cdot 2 = 65.01$; variance $(1 - 0.6^{\mathbf{20}})/(1 - 0.36) = 1.562$
4. The MA root is 1: ΔY_t is not invertible, $\hat{\rho}_{\mathbf{1}} < 0$ and an MA coefficient near -1 are the signs of over-differencing

Interpretation

- A shock dies out: after 10 periods only 0.012 of the deviation is left, and the error variance stays bounded
- The treatment follows the type of trend: detrend a trend-stationary series, difference a series with a unit root (Chapter 3)

Problem 2: identifying Romanian GDP

Series	Terms	ADF τ	p	crit. 1%; 5%; 10%	KPSS	crit. 5%
$100 \ln Y_t$	c, t	-2.31	0.428	-4.05; -3.45; -3.15	0.198	0.146
$100\Delta \ln Y_t$	c	-10.77	< 0.001	-3.49; -2.89; -2.58	0.263	0.463

- **Context**
  - Romanian real GDP, seasonally adjusted (Eurostat), 2000Q1 – 2026Q2
  - Growth $g_t = 100\Delta \ln Y_t$: ACF -0.05, -0.14; PACF -0.05, -0.14 (lags 1, 2; band ± 0.19); Ljung–Box $Q(8) = 7.28$, $p = 0.506$; $\chi^2_{0.95}(8) = 15.51$
- **Tasks**
  1. Describe the ADF test and decide whether $\ln Y_t$ and g_t are stationary.
  2. Say whether KPSS agrees.
  3. Identify the process with the Box–Jenkins method.
  4. The estimated drift is 0.81 (SE 0.30): write the model and interpret the drift.

Problem 2: solution

1. ADF: $\Delta y_t = c + bt + \gamma y_{t-\mathbf{1}} + \sum \delta_j \Delta y_{t-j} + \varepsilon_t$, $H_\mathbf{0}$: $\gamma = 0$ (unit root); $\ln Y_t$: $-2.31 > -3.45$: not rejected; g_t: $-10.77 < -3.49$: rejected
2. KPSS ($H_\mathbf{0}$: stationarity): $0.198 > 0.146$ rejects for the level, $0.263 < 0.463$ does not for growth: $\ln Y_t \sim I(1)$
3. ACF and PACF of g_t inside the band, $Q(8) = 7.28 < 15.51$: growth is white noise around its mean
4. ARIMA(0,1,0) with drift: $100 \ln Y_t = 100 \ln Y_{t-\mathbf{1}} + 0.81 + \varepsilon_t$, $\hat{\sigma} = 2.47$; $t = 2.72$

Interpretation

- The economy grows by about 0.81% a quarter, about 3.2% a year; quarterly surprises are not predictable
- A shock to the level is permanent: the 2009 and 2020 recessions moved the path of GDP for good

Problem 3: SARIMA output and a forecast by hand

Dependent: z_t	Coefficient	SE	z	p
AR(1)	0.353	0.054	6.53	< 0.001
SAR(12)	-0.488	0.035	-13.90	< 0.001
SIGMASQ	0.254			
Ljung–Box $Q(12)$	22.64			0.012

- **Context**
  - $z_t = \Delta\Delta_{\mathbf{12}}\, 100 \ln \mathrm{HICP}_t$ ("DLOG(IPC,1,12)"), Romania, February 2006 – August 2026, $T = 247$
  - Last values: $z = -1.814$ (August 2026), 0.090 (September 2025), 1.794 (August 2025); annual inflation in August 2026: 6.085%
- **Tasks**
  1. Write the equation of the model and interpret the coefficients.
  2. Test whether the residuals are white noise.
  3. Forecast z and the annual inflation rate for September 2026.

Problem 3: solution

1. $(1-\phi L)(1-\Phi L^{\mathbf{12}})z_t = \varepsilon_t$, so $z_t = \phi z_{t-\mathbf{1}} + \Phi z_{t-\mathbf{12}} - \phi\Phi z_{t-\mathbf{13}} + \varepsilon_t$
2. $\hat{\phi} = 0.353$: a monthly surprise continues one month later; $\hat{\Phi} = -0.488 < 0$: a surprise in a month is partly reversed in the same month a year later; both significant; inverted AR roots 0.35 and 0.94 < 1: stationary
3. Ljung–Box $Q(12) = 22.64$ on $12 - 2 = 10$ df, $p = 0.012$: white noise is rejected at 5%; the model leaves some autocorrelation (compare Step 5 above)
4. $\hat{z}_{T+\mathbf{1}} = 0.353(-1.814) + (-0.488)(0.090) - (-0.173)(1.794) = -0.641 + (-0.044) + 0.310 = -0.375$
5. Annual inflation: $\Delta_{\mathbf{12}} \ln P_{T+\mathbf{1}} = \Delta_{\mathbf{12}} \ln P_T + z_{T+\mathbf{1}} = 6.085 + (-0.375) = 5.71\%$

Interpretation

- Inflation is expected to keep falling in September 2026 as last year's tax shock leaves the 12-month window
- Residual autocorrelation: a richer model (as in Step 4) would change the forecast slightly; a 95% interval needs $\hat{\sigma}$

Problem 4: VAR output and Granger causality

ROBOR equation	coef. [t]		coef. [t]
g_{t-1}	0.142 [3.49]	g_{t-2}	0.074 [1.73]
π_{t-1}	0.240 [2.78]	π_{t-2}	−0.172 [−1.89]
i_{t-1}	1.042 [10.16]	i_{t-2}	−0.119 [−1.20]
constant	−0.109 [−0.55]	R^2	0.93

Null hypothesis	F	p
g does not Granger-cause i	7.26	0.001
π does not Granger-cause i	4.74	0.011
i does not Granger-cause π	0.89	0.417

- **Context**
  - VAR(2) for Romania, 2005Q3 – 2026Q2 ($T = 84$): GDP growth g, annual inflation π, ROBOR 3M i
- **Tasks**
  1. Describe Granger causality and interpret the table.
  2. Write the ROBOR equation and quantify the long-run effect of a permanent 1 pp rise in inflation on ROBOR.
  3. Recompute the first F from $RSS_R = 69.268$ and $RSS_U = 58.273$.

Problem 4: solution

1. x Granger-causes y if the lags of x improve the forecast of y given the lags of y (joint F test) Granger (1969)
   - growth and inflation help to forecast ROBOR ($p = 0.001$ and 0.011); ROBOR does not help to forecast inflation ($p = 0.417$)
2. $i_t = -0.109 + 0.142 g_{t-\mathbf{1}} + 0.074 g_{t-\mathbf{2}} + 0.240 \pi_{t-\mathbf{1}} + (-0.172)\pi_{t-\mathbf{2}} + 1.042 i_{t-\mathbf{1}} + (-0.119) i_{t-\mathbf{2}} + u_t$
3. Long run ($i_t = i_{t-\mathbf{1}}$, $\pi_t = \pi_{t-\mathbf{1}}$): $\partial i/\partial \pi = (0.2402 + (-0.1718))/(1 - 1.0424 - (-0.1186)) = 0.0684/0.0762 = 0.90$ pp
4. $F = \frac{(\mathbf{69.268} - \mathbf{58.273})/\mathbf{2}}{\mathbf{58.273}/\mathbf{77}} = 7.26 > F_{\mathbf{0.95}}(2, 77) = 3.12$: $H_{\mathbf{0}}$ rejected

⊡ **Interpretation**

- The money market rate follows the economy and inflation, as a central bank reaction would imply; the reverse link is not visible in quarterly data
- The long-run effect divides by 0.0762, a number close to zero: it is imprecise; Granger causality is predictability, not proof of a policy effect

Problem 5: cointegration and VECM output

Johansen trace test	$r = 0$	$r \leq 1$	$r \leq 2$
statistic	54.60	21.57	3.12
5% critical value	29.80	15.49	3.84

Adjustment α [t]	$\Delta y^{(\mathbf{1})}$	$\Delta y^{(\mathbf{5})}$	$\Delta y^{(\mathbf{10})}$
EC1	-0.021 [-0.82]	0.045 [2.21]	0.040 [2.22]
EC2	0.024 [0.31]	-0.122 [-2.00]	-0.077 [-1.43]

- **Context**
  - US Treasury yields at 1, 5 and 10 years (monthly, FRED), January 1960 – September 2026, VECM with two lagged differences
  - Cointegrating equations: EC1 $= y^{(\mathbf{1})}_{t-\mathbf{1}} - 1.045\, y^{(\mathbf{10})}_{t-\mathbf{1}} + 1.223$; EC2 $= y^{(\mathbf{5})}_{t-\mathbf{1}} - 1.044\, y^{(\mathbf{10})}_{t-\mathbf{1}} + 0.587$
- **Tasks**
  1. Define cointegration and decide the rank.
  2. Write the error-correction part of the equation for $\Delta y^{(\mathbf{10})}_t$ and say which yields adjust.
  3. Interpret the long-run relations economically.

Problem 5: solution

1. $I(1)$ series are cointegrated if a linear combination is $I(0)$ Engle and Granger (1987); trace: $54.60 > 29.80$, $21.57 > 15.49$, $3.12 < 3.84$: rank $r = 2$, one common stochastic trend Johansen (1991)
2. $\Delta y_t^{(\mathbf{10})} = 0.040\,\mathrm{EC1}_{t-\mathbf{1}} + (-0.077)\,\mathrm{EC2}_{t-\mathbf{1}} + \ldots$; on EC1 the 5- and 10-year yields have $|t| > 2$, the 1-year yield $|t| < 1$ on both relations: the short rate does not adjust (weakly exogenous)
3. $\beta \approx 1$: the spreads $y^{(\mathbf{1})} - y^{(\mathbf{10})}$ and $y^{(\mathbf{5})} - y^{(\mathbf{10})}$ are stationary, as the expectations hypothesis implies; half-lives of the equilibrium errors 22.1 and 9.0 months

Interpretation

- The three yields share one trend (the level of rates); the curve can bend, but spreads return to their means
- The long rates do the adjusting: markets move the long end toward the policy-driven short end

Problem 6: GARCH output and VaR 1%

BET	μ	ω	α	β	ν
estimate	0.083	0.0448	0.192	0.799	5.011
SE	0.010	0.0088	0.025	0.024	0.308

Context

- GARCH(1,1) with Student-t innovations for daily BET log returns in %, since 2000 ($n = 6\,670$)
- Today: $\sigma_t^2 = 1.0$ and $\varepsilon_t = -3.0$; $t_5^{-1}(0.01) = -3.3649$; about 250 trading days a year

Tasks

1. Compute the persistence, the half-life and the long-run annual volatility.
2. Compute σ_{t+1}^2.
3. Compute the one-day VaR 1% with the standardised t_5 quantile and compare it with the Normal one.

Problem 6: solution

1. $\alpha + \beta = 0.9907$; $h_{1/2} = \ln 0.5 / \ln 0.9907 = 74$ days; $\bar{\sigma}^2 = 0.0448/(1 - 0.9907) = 4.82$, $\sqrt{250 \times 4.82} = 34.7\%$ a year
2. $\sigma^2_{t+1} = 0.0448 + 0.19162 \times 9 + 0.7991 \times 1.0 = 0.0448 + 1.7246 + 0.7991 = 2.5685$, $\sigma_{t+1} = 1.603\%$
3. $q_{0.01}(z) = -3.3649 \times \sqrt{3/5} = -3.3649 \times 0.7746 = -2.606$; VaR 1% $= -(0.083 + 1.603 \times (-2.606)) = 4.09\%$; Normal: 3.65%

Interpretation

- Persistence close to 1: a shock to volatility halves only after about 74 trading days
- The long-run volatility (34.7%) is far above the sample one (22.1%): when $\alpha + \beta \approx 1$, $\omega/(1 - \alpha - \beta)$ is very imprecise
- The factor $\sqrt{3/5} = \sqrt{(\nu - 2)/\nu}$ rescales the t_5 quantile to innovations of variance 1
- Heavy tails raise VaR 1% above the Normal value: a loss larger than 4.09% is expected on one day in 100

Problem 7: evaluating forecasts

Test month	1	2	3	4	5	6
errors, SARIMA	0.3	−0.5	0.2	0.6	−0.4	0.1
errors, seasonal naive	0.5	−0.7	0.6	0.4	−0.9	0.3

Context

- Monthly inflation forecasts on a test sample of six months; the in-sample MAE of the seasonal naive method is 0.6
- A colleague reports $R^2 = 0.37$ for a boosting model of daily S&P 500 returns, with random 5-fold cross-validation

Tasks

1. Compute RMSE, MAE and MASE for both methods.
2. Compute the Diebold–Mariano statistic on squared errors, $\bar{d}/(s_d/\sqrt{n})$, and decide at 5% ($t_{0.975}(5) = 2.571$).
3. Comment on the colleague's R^2.

Problem 7: solution

1. SARIMA: RMSE 0.389, MAE 0.350, MASE 0.58; seasonal naive: RMSE 0.600, MAE 0.567, MASE 0.94
2. $d_t = e^2_{A,t} - e^2_{B,t}$ (A: SARIMA, B: naive): -0.16, -0.24, -0.32, 0.20, -0.65, -0.08; $\bar{d} = -0.208$, $s_d = 0.281$; DM $= -1.82$, $|-1.82| < 2.571$: equal accuracy is not rejected
3. Random folds put future days in the training set: leakage; walk-forward validation gives $R^2 = -0.39$ for the same model (Chapter 9)

Interpretation

- MASE below 1: SARIMA beats the seasonal naive method on average, but six months cannot show that the gain is real
- Only out-of-sample, time-ordered evaluation counts as evidence of forecasting skill

Problem 8: state space, regimes and long memory

- **Context**
  - Local level model: $\sigma_\varepsilon^2 = 4$, $\sigma_\eta^2 = 1$; prediction $a_t = 10$, $P_t = 2$; observation $y_t = 13$
  - Romanian GDP growth, two regimes: $p_{11} = 0.9108$ (volatile), $p_{22} = 0.9341$ (stable) (Chapter 10)
  - Romanian monthly inflation: ARFIMA(0,d,0) with $\hat{d} = 0.31$ (SE 0.05) (Chapter 8)
- **Tasks**
  1. Run one Kalman update and prediction; compute the steady-state gain and the equivalent SES weight.
  2. Compute the expected durations of the two regimes and the long-run share of the volatile regime.
  3. Compute H and $\rho(1)$ for the ARFIMA model and say whether inflation is stationary and mean-reverting.

Problem 8: solution

1. $F_t = 2+4 = 6$, $K_t = 2/6 = 0.333$; $a_{t|t} = 10 + 0.333 \times 3 = 11.00$, $P_{t|t} = 2(1-0.333) = 1.333$; $a_{t+\mathbf{1}} = 11.00$, $P_{t+\mathbf{1}} = 2.333$
2. $q = \sigma^{\mathbf{2}}_{\eta}/\sigma^{\mathbf{2}}_{\varepsilon} = 0.25$: $\bar{P}/\sigma^{\mathbf{2}}_{\varepsilon} = (q + \sqrt{\mathbf{q^2 + 4q}})/2 = 0.6404$, $\bar{K} = 0.6404/(1+0.6404) = 0.390 = \alpha_{SES}$
3. $E(D_{\mathbf{1}}) = 1/(1-0.9108) = 11.2$ quarters, $E(D_{\mathbf{2}}) = 15.2$ quarters; $\pi_{\mathbf{1}} = (1-0.9341)/(2-0.9108-0.9341) = 0.425$
4. $H = d + 0.5 = 0.81$; $\rho(1) = d/(1-d) = 0.45$; $0 < d < 0.5$: stationary with long memory, mean-reverting

Interpretation

- The filter moves the level one third of the way to the surprise; in the long run the weight is that of exponential smoothing
- In the long run Romanian growth is in the volatile regime about 42% of the time: transition, 2009 and 2020
- Inflation shocks fade slowly, hyperbolically: forecasts return to the mean more slowly than those of an AR model

The team project: content and deliverables

- **20% of the final grade**
  - a team of 2–4 students analyses real series with the methods of the course
  - one concrete question: forecasting EUR/RON, modelling Romanian inflation, monetary policy and bank rates
  - one series: trend and stationarity, smoothing, ARIMA or SARIMA, a training set, a test set and a horizon, point and interval forecasts, two methods compared
  - several series: unit roots, cointegration, VAR or VECM, Granger causality, IRF and FEVD
- **Deliverables**
  - a repository whose code reproduces every number and every chart from the data
  - a short report: the question and the literature, data sources, models, conclusions, references
  - a presentation; the file `AI_USE.md`

Project grading criteria and the oral defence

- **The question**
  - clear, answerable with the data, useful to someone
  - "Can a SARIMA forecast Romanian inflation better than the seasonal naive method?" is a question; "an analysis of inflation" is not
- **Methods, checks, interpretation, reproducibility**
  - the right test for the question (the summary of tests and models); residual diagnostics; out-of-sample evaluation against a benchmark
  - what the numbers mean and what the data cannot show; the repository runs from the data to every chart
- **The oral defence**
  - the project is graded only after its presentation
  - each member explains the code and the results: what does this line compute? why this test and not another?
  - a line of code that cannot be explained does not count as your own work, with or without AI

The file AI_USE.md

- **AI tools are allowed and must be declared**
  - for every use: the tool, the prompt, what was kept and what was corrected
  - the errors of the AI tool that you found and how you found them
- **You are responsible for every number and every reference**
  - every number: recomputed with your own code from the data
  - every reference: its DOI opened and its title checked
- The typical AI errors of this course: the slide “Checks you must run” of the section on AI

The final grade

Component	Weight	When	Criteria
written exam	70%	exam session	method, result, interpretation
team project	20%	end of semester	question, methods, checks, oral defence
attendance and quizzes	10%	every week	presence at the lectures and seminars; the chapter quizzes

- The seminars are for practice and are not graded; students hand in nothing from the seminars
- The quizzes on the course site are taken with the ASE account; name and score are recorded for the grade

Possible contribution of AI

- **Practice**
  - new exercises in the style of the exam, with output generated from real data by your own code
- **Explanation**
  - a second explanation of an output table, a derivation or a test you did not understand
- **Project**
  - a first draft of the code of the Box–Jenkins steps, of a rolling evaluation, of a VECM
- Example prompt
  - `Write Python code that downloads the Romanian HICP from Eurostat (prc_hicp_minr, M.I15.TOTAL.RO), fits every SARIMA(p,1,q)(P,1,Q)12 with p, q <= 2 and P, Q <= 1 to 100 log HICP up to August 2024, reports AICc, BIC and Ljung-Box p-values with m - k degrees of freedom, and compares one-step forecasts with the seasonal naive method by MASE and the Diebold-Mariano test.`

Checks you must run

- The null hypotheses: ADF has a unit root under H_0, KPSS has stationarity; an answer that swaps them is wrong
- The degrees of freedom: Ljung–Box on residuals uses $m - k$; Granger F uses $(p, T - Kp - 1)$
- The signs: the EViews and Python conventions for MA terms and for the error-correction coefficient
- The conventions: AIC and BIC only for the same d and D; VaR 1%, never "VaR 99%"
- The time order: no information from the test sample in the estimation, the scaling or the model choice
- Every number recomputed with your own code; every reference checked by its DOI

Key takeaways

- Look at the data first: components, transformations, the correlogram
- Decide d and D with ADF and KPSS together; identify with ACF and PACF; choose with BIC; check the residuals
- Model the variance when it clusters (GARCH); model several series together with VAR or VECM
- Judge forecasts out of sample against simple benchmarks; a better in-sample fit is not a better forecast
- The exam rewards method, result and interpretation; the project rewards a clear question answered honestly

Self-assessment

- **Question**: ADF does not reject and KPSS rejects. What do you conclude?
  - **Answer**: both point to a unit root: difference the series
- **Question**: the residuals of an ARMA(1,1) give Ljung–Box $Q(10) = 16$. Is the model adequate at 5%?
  - **Answer**: with $10 - 2 = 8$ df the critical value is 15.51: no, autocorrelation is left
- **Question**: two $I(1)$ series are cointegrated. Why not a VAR in differences?
  - **Answer**: it omits the error-correction term and loses the long-run information

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